Diffraction from Impedance Loaded Rectangular Structures

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Abstract

Employing high frequency approximations, Rawlins [1] developed some simple formulae for the scattering of an E-polarised plane wave by an infinite rectangular cylindrical structure, on which an impedance boundary condition is enforced. An integral equation approach for which a rapidly convergent quadrature scheme is described. In order to validate Rawlins' solutions, the computed scattered far-field patterns are compared with the results of [1] for rectangular structures of varying aspect ratio and over a range of impedances.

1. Introduction

The electromagnetic wave propagation of radio, television and mobile phone signals in cities is influenced by building corners and their surface cladding. In [1] Rawlins discusses the important factors for the signal strength of phones in such environments and develops an idealised model relevant to this scenario: the diffraction of an E-polarised wave by an absorbing rectangular cylinder. It utilises Keller's method of GTD and its extensions to deal with multiple diffraction, and employs the diffraction coefficient derived for the canonical problem of diffraction by an impedance wedge (specialized to a right-angled corner) that had already been derived in an earlier paper [2]. If θ_0 and θ denote the angle of incidence and the angle of observation from the same planar face, the far-field expression for the diffracted at distance r takes the form (k denoting wavenumber)

$$u_d\,(r,\theta,\theta_0) = \frac{D(\theta,\theta_0)e^{ikr}}{\sqrt{r}} + O\big((kr)^{-3/2}\big)$$

as $kr \to \infty$, where the diffraction coefficient $D(\theta,\theta_0)$ given by Rawlins [1]. By adding together the diffracted ray contributions from the four corners, relatively simple high frequency approximate expressions for the scattered far-field resulting from a plane wave obliquely incident on an imperfectly conducting rectangle can be obtained; this approximation takes into account the contribution of simple diffraction by each corner. The approximation is improved by including the effects of multiple diffraction by the corners; in [1], the expression for the scattered far-field taking into account double diffraction effects is obtained.

In this paper, we undertake a numerical study of the scattering of an E-polarised plane wave by an infinite cylindrical structure on which an impedance boundary condition is enforced at all points on the cross-sectional boundary of the cylinder. Following the approach of Colton and Kress [3] we employ an integral equation for the unknown surface distribution comprising a single-layer potential and the adjoint of the double-layer potential. A Nystrom method similar to that expounded by Colton and Kress [4] (for the soft boundary condition) is developed for the numerical solution of this integral equation. The computed scattered far-field is compared with the results of Rawlins [1] in order to validate his solutions over a range of impedances and varying aspect ratios.

2. An integral equation approach

An infinitely long cylinder with uniform cross-section, and axis parallel to the z-axis, and is illuminated by an E- polarised incident plane wave propagating with direction parallel to the x-y plane. We will assume that the cross-section D lying in the x-y plane has a smooth closed boundary ∂D that can be parameterised by $\boldsymbol{x}(t) = \big(x_1\,(t), x_2\,(t)\big)$ for $t \in [0,2\pi]$. We assume that the incident and scattered fields are time harmonic with a temporal factor $e^{i\omega t}$. The spatial component $u^{sc}(x,y)$ of the scattered field obeys the Helmholtz equation

$$(\Delta + k^2)u^{sc}(x,y) = 0$$

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at all points (x, y) exterior to the body, where $k = \omega/c$ is the wavenumber and c the speed of light in free space; moreover it obeys the two-dimensional form of the Sommerfeld radiation condition [4]. The spatial component of the incident wave travelling in the direction of the unit vector $\boldsymbol{d} = (\cos\theta_0, \sin\theta_0)$ takes the form $u^{inc}(x, y) = e^{ik\boldsymbol{x}.\boldsymbol{d}}$.

The following impedance boundary condition is enforced on the total field $u^{tot}(x, y) = u^{inc}(x, y) + u^{sc}(x, y)$ at all points $\boldsymbol{x} = (x, y)$ on the boundary ∂D,

$$\frac{\partial u^{tot}(x, y)}{\partial n(\boldsymbol{x})} + ik\lambda u^{tot}(x, y) = 0,$$

where $n(\boldsymbol{x})$ is the unit outward normal to the boundary at the point $\boldsymbol{x}$ and $\lambda = \lambda(\boldsymbol{x})$ is a continuous function of position. The scattered field is uniquely determined by the boundary and radiation conditions, provided $Re(\lambda)$ is positive on the boundary ∂D. In this paper, λ will be restricted to be a (complex) constant.

As shown in [4], the scattered field may be determined by employing the single-and double-layer potentials associated with the two dimensional free-space Green's function

$$\mathrm{G}(\mathbf{x}, \mathbf{y}) = \frac{i}{4} H_0^{(1)}(k|\mathbf{x} - \mathbf{y}|),$$

where $H_0^{(1)}$ denotes the Hankel function of first kind and order zero. Define two operators associated with the single- and double-layer potentials of a continuous density $\varphi(\boldsymbol{y})$ defined on the boundary ∂D, namely,

$$S\varphi(\boldsymbol{x}) = 2\int_{\partial D} G(\boldsymbol{x}, \boldsymbol{y})\ \varphi(\boldsymbol{y}) ds(\boldsymbol{y})$$

and

$$K'\varphi(\boldsymbol{x}) = 2\int_{\partial D} \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial n(\boldsymbol{x})}\ \varphi(\boldsymbol{y}) ds(\boldsymbol{y})$$

Then the single layer potential $\frac{1}{2}S\varphi(\boldsymbol{x})$ provides a solution, at all points $\boldsymbol{x}$ exterior to the body D, to the exterior impedance problem formulated above provided the continuous density $\varphi(\boldsymbol{x})$ is a solution to the following integral equation on ∂D:

$$\varphi - K'\varphi - ik\lambda S\varphi = -2g \qquad (1)$$

where $g(\boldsymbol{x}) = -(\frac{\partial u^{inc}(\boldsymbol{x})}{\partial n(\boldsymbol{x})} + ik\lambda u^{inc}(\boldsymbol{x}))$.

The solution is unique provided k is not an interior Dirichlet eigenvalue (i.e. the Helmholtz equation with soft boundary conditions does not support non-trivial solutions in the interior of D).

3. Numerical Solution

The integral equation (1) forms the basis for our solution. With the given parameterisation for ∂D, the outward pointing unit normal at $\boldsymbol{x} = \boldsymbol{x}(\tau)$ is

$$n(\tau) = (x_2'(\tau), -x_1'(\tau))/J(\tau)$$

where $J(\tau)$ is the Jacobian factor

$$J(\tau) = \sqrt{(x_1'(\tau))^2 + (x_2'(\tau))^2}.$$

The operators may then be expressed as

$$S\varphi(\boldsymbol{x}(t)) = \int_0^{2\pi} S_0(t, \tau)\ \varphi(\tau) d\tau, \qquad K'\varphi(\boldsymbol{x}(t)) = \int_0^{2\pi} K'_0(t, \tau)\ \varphi(\tau) d\tau,$$

where $\varphi(\tau) = \varphi(\boldsymbol{x}(\tau))$, and

$$S_0(t, \tau) = 2G(\boldsymbol{x}(t), \boldsymbol{x}(\tau))J(\tau), \qquad K_0'^{(t,\tau)} = 2\frac{\partial}{\partial n(\boldsymbol{x}(t))} G(\boldsymbol{x}(t), \boldsymbol{x}(\tau))J(\tau).$$

The kernels have a logarithmic singularity at $t = \tau$.

A suitable numerical method for the solution of the integral equation (1) is described in [4]; it employs a quadrature rule especially adapted to the logarithmic singularity. Now choose $2n$ uniformly spaced points $t_j = \frac{\pi j}{n}$, for $j = 0, 1, \ldots, 2n - 1$. With these quadrature rules evaluated at the $2n$ points t_j we have obtained a system of $2n$ linear equations for the boundary values $\varphi(t_j)$ $(j = 0, 1, \ldots, 2n - 1)$ that are obtained by the usual Gaussian elimination procedure.

The numerical results discussed in the next section were obtained after implementation of this scheme in a MATLAB code. A number of tests were applied to verify its correctness: a comparison with the analytic Mie series solution for a circular boundary; for non-circular scatterers, examination of the convergence rate as a function of n was observed to be exponentially fast ('super-algebraic), as expected [4]; also for such scatterers, a comparison was made with the exact solution obtained due to placement of an artificial point source at an interior point. The condition number of the system was checked to ensure that uniqueness problems arising for wavenumbers k near an interior Dirichlet eigenvalue of the scatterer were avoided.

3. Results and discussion

The so-called super–ellipse is defined in the x-y plane by

$$(\frac{x}{a})^m + (\frac{y}{b})^m = 1,$$

where a and b are positive; we take m to be an even integer. It may be parameterised in terms of a parameter t by setting $y = x\tan(t)$ so that

$$x = \frac{\cos t}{(cos^m t/a^m + sin^m t/b^m)^{1/m}}, \qquad y = \frac{\sin t}{(cos^m t/a^m + sin^m t/b^m)^{1/m}}.$$

As m is increased the figure approximates a rectangle of sides $2a$ and $2b$ with rounded corners. This figure is used in our calculations and provides the basis of our comparison with, and validation of, Rawlins' results [1]. Far field patterns and other field data obtained below were computed with increasingly large values of m until no discernible change to the pattern was observed; typically $m = 32$ sufficed. Of course, although the pattern barely changes, an increasing number of quadrature points is needed to obtain good convergence for the structure with an increasingly sharp corner.

Figure 1 shows the far field pattern for a plane wave incident at angle $\theta_0 = 3\pi/4$ with wavenumber $k = 2\pi$ on a rounded rectangle with sides $a = 1, b = 2$ and impedance parameter $\lambda = 1 + i$. When the parameter m is chosen sufficiently large, the results are in close agreement with those obtained by Rawlins [1, Fig. 20], especially in the direction of the main and side lobes and in their relative magnitudes, as well as much of the fine detail. The main lobe is aligned with the direction of the incident field.

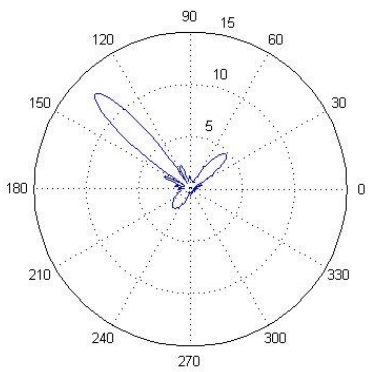

Figure 1: rectangular scatterer (a=1, b=2), k=2π, $\lambda = 1 + i, \theta_0 = 3\pi/4$.

The far-field patterns for the other rectangular structures examined by Rawlins [1] were computed; in these cases, $a = 1$ and $b = 1\ or\ 2$, and the impedance parameter takes the values $\lambda = 1 + i, 4 + i, 1 + 10i\ or\ 10 + i$; there is close agreement in each case (see [5]). (In particular, the far-field pattern of a rectangular scatterer with large impedance parameter approaches, as expected that of the corresponding soft scatterer.)

It is of interest to examine the scattering by the rectangle as the aspect ratio b:a increases. With $a = 1, b = 6$ and $\lambda = 1 + 10i$ (and $k = 2\pi$), the far-field pattern computed by the integral equation approach differs significantly from that computed by the double diffraction approach. Multiple checks have been carried out to validate the computations. The reasons for the divergence of results will be discussed, including the adequacy of terms in the double diffracted approach.

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5. References

1. A. D. Rawlins, "High frequency diffraction of electromagnetic plane wave by an imperfectly conducting rectangular cylinder", *J. Eng. Math*. (2012) **76**:157-180.

2. A. D. Rawlins, "Asymptotics of a right-angled impedance wedge", *J. Eng. Math*. (2019) **65**:355-366.

3. D.L. Colton and R. Kress, *Integral equation methods in scattering theory*, Wiley (New York), 1983.

4. D.L. Colton and R. Kress, *Inverse Acoustic and Electromagnetic Scattering Theory*, 3rd Ed., Springer-Verlag (New York), 2013.

5. P. D. Smith and A. D. Rawlins, "Diffraction from structures with an impedance boundary", International Conference on Electromagnetics in Advanced Applications, Torino, Italy, 2013, pp. 1036-1039.