Generalized Natural Mode Expansion for Arbitrary Electromagnetic Fields

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Abstract

A generalized natural mode expansion theory for investigating arbitrary electromagnetic fields is presented. When an inhomogeneity is bounded with impenetrable boundaries, the field excited by arbitrary sources is expanded with a complete set of eigenmodes, which are classified into trapped modes and exterior modes. As the boundaries tend to infinity, trapped modes remain unchanged, while exterior modes form a continuum. In numerical studies, unbounded systems can be emulated by placing perfectly matched layers (PMLs) at finite extent. Simulation shows that only a few natural modes are prominent in expanding the excited field in a properly functioning device. Such a reduced modal picture can provide quick guidance as well as useful physical insight for engineering design and optimization of electromagnetic components and devices.

1 Introduction

Eigenmode expansion theory has been widely used in analyzing electromagnetic (EM) problems. Many previous studies on eigenmode expansion have a limitation that they are applicable to specific bounded systems, such as waveguides and cavities [1]. We present a generalized natural mode expansion analysis, which leads to a unified treatment for both bounded and unbounded media.

We start with a bounded EM problem where the governing operator in wave equation is symmetric or self-adjoint. We seek the excited field due to any source with a set of complete and orthogonal eigenmodes, which are also known as the natural modes of the system. When the governing operator is nonsymmetric or non-self-adjoint, an auxiliary problem is formed with respect to the original one, and a set of bi-orthogonal eigenmodes can be used to expand arbitrary excited fields. The unbounded problem is approached by moving the boundaries to infinity. By doing so, modes forming a continuum are considered as exterior modes, while those which remain invariant are trapped modes. Furthermore, model order reduction is available for general excitation problems where only several natural modes dominate. Hence, it is possible to offer physical insight into the working of many electromagnetic structures, such as microstrip antennas, small antennas, and metamaterial-inspired structures. It is worthy to note that the proposed reduced modal analysis is field-centered, which is achieved with a finite-difference or finite-element based discretization. It is closely related to the current-centered singularity expansion method [2], and differs from the characteristic mode analysis [3] where characteristic modes can never be physical natural modes.

2 Formulation

2.1 Bounded Problem

We consider a linear inhomogeneity in domain V enclosed by surface S. When sources exist in V, the time-harmonic vector wave equation for electric field is

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}(\mathbf{r}) - \omega^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}(\mathbf{r}) = \mathbf{S}(\mathbf{r}). \tag{1}$$

where $\mathbf{S}(\mathbf{r}) = i\omega\mathbf{J}(\mathbf{r}) - \nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \mathbf{M}(\mathbf{r})$, and $\mathbf{J}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$ are electric and magnetic current sources, respectively. The field on S satisfies $\hat{n} \times \mathbf{E}(\mathbf{r}) = 0$ or $\hat{n} \times \mathbf{H}(\mathbf{r}) = 0$, where $\hat{n}$ is the unit normal vector of S.

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The source-free problem is essentially a linear eigenvalue problem. The natural modes are countably infinite eigen-solutions $\mathbf{E}_n(\mathbf{r})$ for certain discrete values of ω_n^2 described by

$$\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla \times \mathbf{E}_n(\mathbf{r}) - \omega_n^2 \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n(\mathbf{r}) = 0. \tag{2}$$

When the medium is lossless, namely $\overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\epsilon}}^\dagger$ and $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^\dagger$, or reciprocal, namely $\overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\epsilon}}^{\mathrm{t}}$ and $\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}^{\mathrm{t}}$, one can prove that both operators $\nabla \times \overline{\boldsymbol{\mu}}^{-1} \cdot \nabla\times$ and $\overline{\boldsymbol{\epsilon}}\cdot$ are self-adjoint or symmetric under PEC/PMC boundaries. Hence, for $\omega_m^2 \neq \omega_n^2$, we have, with the defined inner products below,

$$\langle \mathbf{E}_m^\gamma(\mathbf{r}), \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n(\mathbf{r}) \rangle = \int_V \mathbf{E}_m^\gamma(\mathbf{r}) \cdot \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n(\mathbf{r}) \, \mathrm{d}\mathbf{r} = 0, \tag{3}$$

where $\gamma = *$ for the lossless case, and omitted for the reciprocal case. Nontrivial degenerate eigenmodes satisfying $\omega_m^2 = \omega_n^2 \neq 0$ can be orthogonalized by applying the Gram-Schmit process. Note that the curl-curl operator in (1) has a null space spanned by countably infinite irrotational eigenmodes, which shall be included in low-frequency applications, or in problems where the sources are not divergence free.

Using natural modes $\mathbf{E}_n(\mathbf{r})$, the solution of (1) in V can be expanded as

$$\mathbf{E}(\mathbf{r}) = \sum_n^\infty \alpha_n \mathbf{E}_n(\mathbf{r}) \tag{4}$$

with coefficients α_n calculated as

$$\alpha_n = \frac{1}{\omega_n^2 - \omega^2} \frac{\langle \mathbf{E}_n^\gamma, \mathbf{S} \rangle}{\langle \mathbf{E}_n^\gamma, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n \rangle} \tag{5}$$

where γ is chosen appropriately for the lossless or reciprocal case.

When the inhomogeneity is lossy or non-reciprocal, (3) is no longer valid. Solving for α_n may not be as trivial as in (5) since a matrix inversion is needed after testing (1). A remedy for this difficulty is to introduce an auxiliary system to the original one. In the lossy case, the auxiliary equation of (2) is constructed as [4]

$$\nabla \times \left[\overline{\boldsymbol{\mu}}^{-1}\right]^\dagger \cdot \nabla \times \mathbf{E}_n^{\mathrm{a}}(\mathbf{r}) = \left(\omega_n^2\right)^* \cdot \overline{\boldsymbol{\epsilon}}^\dagger \cdot \mathbf{E}_n^{\mathrm{a}}(\mathbf{r}) \tag{6}$$

while in the non-reciprocal case, the auxiliary equation is

$$\nabla \times \left[\overline{\boldsymbol{\mu}}^{-1}\right]^{\mathrm{t}} \cdot \nabla \times \mathbf{E}_n^{\mathrm{a}}(\mathbf{r}) = \omega_n^2 \cdot \overline{\boldsymbol{\epsilon}}^{\mathrm{t}} \cdot \mathbf{E}_n^{\mathrm{a}}(\mathbf{r}) \tag{7}$$

Both (6) and (7) should be constructed in the same domain V as in (2), and the auxiliary fields, denoted by superscript a, should satisfy the same boundary conditions as in (2) but with different material media. Therefore, we can obtain the bi-orthogonality conditions ($\omega_m^2 \neq \omega_n^2$) as

$$\langle \mathbf{E}_m^{\mathrm{a}\gamma}, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n \rangle = 0. \tag{8}$$

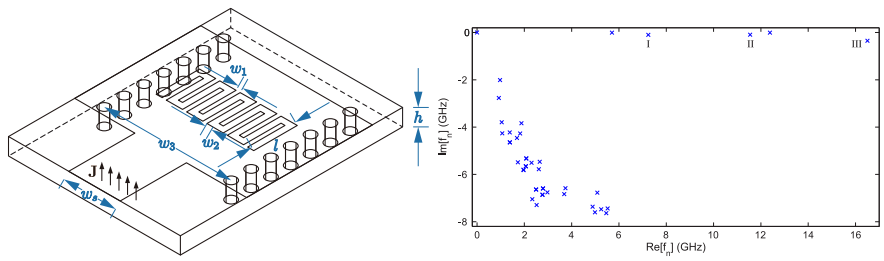
Hence, the field solution due to any source excitation $\mathbf{S}(\mathbf{r})$ can be expanded in terms of the complete set (assumed to be) of eigenbasis $\mathbf{E}_n(\mathbf{r})$ where α_n can be calculated in a simple form of

$$\alpha_n = \frac{1}{\omega_n^2 - \omega^2} \frac{\langle \mathbf{E}_n^{\mathrm{a}\gamma}, \mathbf{S} \rangle}{\langle \mathbf{E}_n^{\mathrm{a}\gamma}, \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E}_n \rangle}. \tag{9}$$

2.2 Unbounded Problem

The extension of the generalized theory to an unbounded problem is realized by letting S tend to infinity, where the natural mode expansion can be applied. Meanwhile, a small loss has to be introduced to the medium in V, which is equivalent to requiring the wave to be outgoing at infinity. Besides, such a small loss guarantees the uniqueness of the field solution due to some given sources.

In an unbounded medium, the natural mode expansion (4) can be written in a form with more physical insight if natural modes are classified into trapped modes and exterior (radiation) modes. The former resonate



(a) (b)

Figure 1: (a) Open-ended CRLH-SIW slot antenna. (b) Computed complex eigenfrequencies close to 11 GHz.

due to the inhomogeneity with most energy confined, while the latter resonate between the inhomogeneity and the far boundaries. Trapped modes may couple to external radiation in some systems, which are also regarded as leaky modes. By letting boundaries approach infinity, the spectrum of trapped modes slightly broadens, and the modal shapes remain almost unchanged. However, exterior modes are not 'immune' to external variations, and their number becomes denser and denser as boundaries expand, which eventually yields a continuum. Thus, we can intuitively write the expanded solution of unbounded fields as

$$\mathbf{E}(\mathbf{r}) = \sum_{n=1}^{N} \alpha_n \mathbf{E}_n(\mathbf{r}) + \int \mathrm{d}\omega' \alpha(\omega') \mathbf{E}(\omega', \mathbf{r}). \tag{10}$$

The summation and integral correspond to trapped modes and exterior modes, respectively. Singularity is avoided due to the small loss introduced to the system. Also of note is that α_n are non-dimensional expansion coefficients, and $\alpha(\omega')$ involve the systems density of states (DOS) with dimension of ω^{-1}.

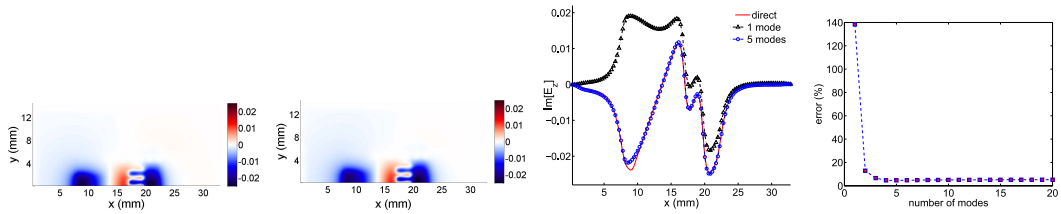
Another strategy to emulate the unbounded problem is to replace PEC or PMC boundaries with PMLs [5, 6] at finite extent. This has been applied to a few 1-D and 2-D guiding problems [7]. When PMLs are implemented, the eigen-spectrum becomes discrete from which trapped modes can be easily distinguished from exterior ones since the quality factors of the former are obviously larger. Moreover, exterior modes in this case include resonances inside PMLs which have large $\Im m[\omega_n]$. Using PMLs enables one to correctly capture the physics of trapped modes which are important in characterizing the field behavior of an excited finite system. Therefore, the model order can be greatly reduced through natural mode expansion using only a few important eigenmodes.

3 Numerical Result

A locally-conformal finite-difference scheme is used to analyze a substrate integrated waveguide based antenna, which can operate below waveguide cutoff frequency [8] by incorporating a composite right/left-handed slot structure [Fig. 1(a)]. The dimension of the antenna is given in [9]. We search for 50 natural modes whose eigenfrequencies are close to 11 GHz. Six-layer PMLs are used to emulate the unbounded problem. Implicitly restarted Arnoldi method (IRAM) is used as the eigensolver, and inverse operation for shift-and-invert in IRAM is achieved by using the efficient unsymmetric multifrontal method implemented in UMFPACK.

The computed complex eigenfrequencies are plotted in Fig. 1(b), where 5 trapped modes with small $\Im m[f_n]$ indicating large Q factors are easily identified. Here, modes I, II and III at $7.24 - 0.01i$ GHz, $11.55 - 0.01i$ GHz and $16.51 - 0.035i$ GHz shall correspond to the -1st, 0th and 1st order resonances, respectively. The other trapped modes correspond to slot resonances that are hard to be excited. The rest with large $\Im m[f_n]$ are exterior modes or PML resonances. To validate the results, we obtain the S_{11} response of a similar structure by HFSS with the corresponding dips at 7.23 GHz, 11.20 GHz, and 16.88 GHz, respectively, which are in consistency with the computed $\Re e[f_n]$ for modes I, II and III.

A z-polarized current sheet operating at 5.9 GHz is introduced to the aforementioned slot antenna [Fig. 1]. As shown in Figs. 2(a) and (b), the excited field $\Im m[E_z^{\mathrm{dir}}]$ obtained by directly solving Equation (1) agrees well with the expanded field $\Im m[E_z^{\mathrm{exp}}]$ using the trapped modes. Hence, a reduced modal expansion is achieved. More quantitatively, comparisons of the directly calculated field and the expanded fields with



(a) (b) (c) (d)

Figure 2: $\Im m[E_z]$ of: (a) Directly calculated field on xy plane at $z = 0.635$ mm. (b) Expanded field using trapped modes on the same plane. (c) Comparisons of directly calculated and expanded $\Im m[E_z]$ along an observation line $y = 0.36$ mm. (d) Relative error with respect to the number of modes for expansion.

different numbers of trapped modes along an arbitrary observation line, e.g., $y = 0.36$ mm, are given in Fig. 2(c). The relative error, calculated as $\left(\frac{\|\Im m[E_z^{\text{exp}} - E_z^{\text{dir}}]\|_{\text{F}}}{\|\Im m[E_z^{\text{dir}}]\|_{\text{F}}}\right)$ where $\|\cdot\|_{\text{F}}$ is the Frobenius norm, is shown in Fig. 2(d) with respect to the number of modes used to expand the field.

4 Conclusions

A generalized natural mode expansion is presented to analyze complicated bounded and unbounded electromagnetic systems. A reduced modal picture is demonstrated to offer useful physical insights and guidance in microwave and antenna design and optimization.

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