

Hierarchical Bases on the Standard and Dual Graph for Stable Solutions of the EFIE Operator

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Abstract

This paper presents a dual generalized Haar basis for the electric field integral equation (EFIE) that regularizes the vector and the scalar potential on structured and unstructured meshes. Hierarchical preconditioners that regularize both potentials of the EFIE operator have been developed for structured meshes (for example obtained by a dyadic mesh refinement), but not for unstructured ones. In this contribution, we leverage graph Laplacians to transform the scalar potential into a single layer potential, while the vector potential is first transformed into the hypersingular operator and then into an operator that is equivalent to the single layer potential up to a compact perturbation by using the inverse Laplace-Beltrami operator. Then generalized Haar bases constructed from graph Laplacians of the primal and dual mesh are applied. Notice that the new preconditioner maintains the leading complexity set by fast matrix-vector multiplication methods. The presented results demonstrated the validity and effectiveness of the proposed approach and highlight the necessity thereof.

1 Introduction

Hierarchical basis preconditioners have been developed and employed to cure the low-frequency and the dense-discretization breakdown of the electric field integral equation (EFIE) operator. While the low-frequency breakdown is a simple matter of ill-scaling between the vector and scalar potential (an ill-scaling that can be cured by using a quasi-Helmholtz decomposition with subsequent rescaling), the dense-discretization breakdown is intrinsically linked to integrative and derivative nature of the vector and scalar potential, respectively. Using a hierarchical basis allows to differentiate between the weakly and strongly oscillating eigenmodes of the vector and scalar potential on the surface, and by applying a diagonal preconditioner the ill-conditioning because of the decreasing (vector potential) and increasing (scalar potentials) eigenvalues is annihilated.

The first hierarchical basis preconditioners for the EFIE were developed only for the scalar potential and only for scenarios where structured meshes are involved [1]. In such a scenario, the mesh for which the problem has to be solved is obtained by refining, for example, dyadically, a coarser mesh until the desired fineness of the mesh is achieved. Applying a hierarchical basis leads to a system matrix, where the condition number only grows with $\mathcal{O}(1/h)$. When a more recent hierarchical basis is applied, also the vector potential is regularized. In this case, the condition number grows logarithmically with the number of unknowns and the obtained condition number is in the same order as the condition number of the system matrix stemming from the discretization on the coarse mesh [2].

However, the applicability of such a basis is limited to structured meshes. It comes to no surprise that efforts have been made to obtain hierarchical bases also for the standard situation when unstructured meshes must be used. Such bases can be generated by using a clustering scheme with which macro basis functions can be defined, but so far only for the scalar potential these bases have been developed and successfully applied [3].

It might seem odd that there is no hierarchical basis available for the vector potential on unstructured meshes, while we have such bases for the scalar potential. To understand this, one must recall that in the charge space the scalar potential is merely a single layer potential. For the single layer potential a hierarchical basis that can represent the space of patch basis functions can easily be constructed, and the only challenge is to find vector functions whose divergences equals these hierarchical basis functions.

Yet, there is hope for a hierarchical regularization of the vector potential on unstructured meshes, and this hope is “duality”. Duality is a principle that permeates electromagnetics. Many recent technologies in electromagnetics have been derived by dual principles, take for example the Buffa-Christiansen functions, which are dual to the Rao-Wilton-Glisson functions. In fact, we can construct a generalized Haar basis (a hierarchical basis that is constructed from patch basis function space) for the primal and the dual mesh. While for the primal mesh we use the cell-based graph Laplacian, we use the node-based graph Laplacian for the dual mesh. Lamentably, a generalized Haar basis can only be applied to the single layer potential (or an operator that is equivalent to the single layer potential up to a compact perturbation).

The approach, we investigate in this paper, is twofold. For the scalar potential, we employ inverse graph Laplacians to transform it into the single layer potential of the charge space. The vector potential, instead, is first transformed into the hypersingular operator by using the loop basis functions. Then by chaining the hypersingular operator twice with itself and the inverse Laplace-Beltrami operator, we obtain an operator equivalent to the single layer potential up to a compact operator. Numerical results demonstrated the effectiveness and relevance of the proposed approach. Preliminary results have appeared in a conference contribution [4].

2 Background

Let Γ be an open or closed, perfectly electrically conducting (PEC) surface situated in the free space. An electromagnetic wave described by the electric \mathbf{E}^i and magnetic \mathbf{H}^i impinges on the surface. The impinging wave generates a surface current density \mathbf{j} , which generates a scattered wave described by the fields $(\mathbf{E}^s, \mathbf{H}^s)$. On the surface, the tangential component of scattered electric field

$$\hat{\mathbf{n}} \times \mathbf{E}^s = \mathcal{T}(\mathbf{j}) = \hat{\mathbf{n}} \times \mathcal{S}(\mathbf{j}) = \hat{\mathbf{n}} \times ik \int_{\Gamma} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \mathbf{j}(\mathbf{r}') d\mathbf{r}' - \hat{\mathbf{n}} \times \underbrace{\frac{1}{ik} \nabla \int_{\Gamma} \frac{e^{ik|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \nabla \cdot \mathbf{j}(\mathbf{r}') d\mathbf{r}'}_{\text{Scalar potential}}. \quad (1)$$

As the tangential component of the total electric field must vanish on the surface of a PEC scatterer, we have $\hat{\mathbf{n}} \times \mathbf{E}^s = -\hat{\mathbf{n}} \times \mathbf{E}^i$. We then find the EFIE: $-\hat{\mathbf{n}} \times \mathbf{E}^i = \mathcal{T}(\mathbf{j})$. In order to find \mathbf{j} , we approximate the surface Γ with a planar triangulation $\tilde{\Gamma}$ and discretize the EFIE using a Galerkin approach. We employ as test and basis functions the Rao-Wilton-Glisson (RWG) functions [5] resulting in the linear system of equations

$$\mathbf{Z}\mathbf{i} = \mathbf{v}, \quad (2)$$

where the system matrix $[\mathbf{Z}]_{ij} = (\mathbf{f}_i, \mathcal{S}(\mathbf{f}_j))_{L^2(\tilde{\Gamma})}$, the right-hand side excitation $[\mathbf{v}]_i = -(\mathbf{f}_i, \mathbf{E}^i)_{L^2(\tilde{\Gamma})}$, and \mathbf{i} is the unknown current with $\mathbf{j} \approx \sum_{j=1}^N \mathbf{f}_j [i]_j$, where $N = \dim V^{\text{RWG}}$ is the number of unknowns; following standard conventions, we use

$$(\mathbf{f}(\mathbf{r}), \mathbf{g}(\mathbf{r}))_{L^2(\tilde{\Gamma})} = \int_{\tilde{\Gamma}} \mathbf{f} \cdot \mathbf{g} dS(\mathbf{r}). \quad (3)$$

The EFIE operator \mathcal{T} is a vector-valued operator. Developing a hierarchical basis for a vector-valued functions is more complex than for a scalar-valued operator. By leveraging a quasi Helmholtz-decomposition, we transform the EFIE operator in such a way that eventually we have to deal only with two scalar integral operators: the single layer potential and the hypersingular operator. Let $\mathbf{\Lambda}$ be the transformation matrix that maps solenoidal loop functions to RWG functions and let $\mathbf{\Sigma}$ be the transformation matrix that maps non-solenoidal star functions to RWG functions. Then we find

$$\begin{bmatrix} \mathbf{\Lambda}^T \\ \mathbf{S}^T \end{bmatrix} \mathbf{Z} \begin{bmatrix} \mathbf{\Lambda} & \mathbf{S} \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{\Lambda\Lambda} & \mathbf{Z}_{\Lambda\Sigma} \\ \mathbf{Z}_{\Sigma\Lambda} & \mathbf{Z}_{\Sigma\Sigma} \end{bmatrix} = \mathbf{Z}_{\text{qH}}. \quad (4)$$

It can be shown that the matrices $\mathbf{Z}_{\Lambda\Lambda}$ and $\mathbf{Z}_{\Sigma\Sigma}$ can be expressed in the static limit as $\lim_{k \rightarrow 0} 1/k [\mathbf{Z}_{\Lambda\Lambda}]_{ij} = (\lambda_i, \mathcal{W}\lambda_j)_{L^2}$ and $\lim_{k \rightarrow 0} k [\mathbf{Z}_{\Sigma\Sigma}]_{ij} = (\phi_i, \mathcal{V}\phi_j)_{L^2}$, where

$$\mathcal{W}\lambda_i = \hat{\mathbf{n}} \cdot \nabla \times \int_{\tilde{\Gamma}} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \nabla \times \hat{\mathbf{n}}_{\mathbf{r}'} \lambda_i(\mathbf{r}') dS(\mathbf{r}') \quad (5)$$

is the hypersingular operator with λ_i being the linear Lagrangian function defined on node i and

$$\mathcal{V}\phi_i = \int_{\tilde{\Gamma}} \frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|} \phi_i(\mathbf{r}') dS(\mathbf{r}') \quad (6)$$

is the single layer potential with $\phi_i = \text{div}_{\tilde{\Gamma}} \mathbf{\Sigma}_i$, where $\mathbf{\Sigma}_i$ is the star function defined on cell i . As the divergence operation is given by $\mathbf{\Sigma}^T$, we find the relationship

$$\lim_{k \rightarrow 0} k \mathbf{Z}_{\Sigma\Sigma} = \mathbf{\Sigma} \mathbf{V} \mathbf{\Sigma}^T, \quad (7)$$

where \mathbf{V} is the system matrix of the single layer potential when \mathcal{V} is discretized with patch basis functions.

3 New Hierarchical Preconditioner

Relationship (7) comes in handy. We are looking for a hierarchical basis that represents the patch function space. Thus desire a linear, sparse mapping \mathbf{H}_Σ from the hierarchical functions to the patch functions. Instead of directly discretizing the single layer potential with the hierarchical basis, we can simply use $\mathbf{H}_\Sigma^\top \mathbf{V} \mathbf{H}_\Sigma$. We call such a basis generalized Haar basis. Because of the space limitation, we do not detail the construction of the generalized Haar basis in this contribution. However, the reader should be aware that such a basis can be constructed with $\mathcal{O}(N \log N)$ complexity, that \mathbf{H}_Σ is sparse, and that if we use a spectral domain decomposition approach to generate the hierarchy, the only input required is a cell-based graph Laplacian.

The matrix $\mathbf{H}_\Sigma^\top \mathbf{V} \mathbf{H}_\Sigma$ is not yet well-conditioned, but $\mathbf{D}_\Sigma \mathbf{H}_\Sigma^\top \mathbf{V} \mathbf{H}_\Sigma \mathbf{D}_\Sigma$ is well-conditioned when \mathbf{D}_Σ is a diagonal preconditioner. In fact, the condition number grows only logarithmically in the number of unknowns [6]. The diagonal preconditioner can be obtained in linear complexity by using only the strong interaction of the single layer potential and by performing the sparse matrix products explicitly.

As the right-inverse of Σ^\top is given by $\Sigma (\Sigma^\top \Sigma)^\dagger$, where the \dagger symbol denotes the Moore-Penrose pseudoinverse, we find $\lim_{k \rightarrow 0} k (\Sigma^\top \Sigma)^\dagger \Sigma^\top \mathbf{Z}_{\Sigma\Sigma} \Sigma (\Sigma^\top \Sigma)^\dagger = \mathbf{V}$. Therefore, the matrix $\mathbf{S} = \Sigma (\Sigma^\top \Sigma)^\dagger \mathbf{H}_\Sigma$ is a valid hierarchical preconditioner for the scalar potential. Two remarks concerning the implementation of that preconditioner: the matrix $\Sigma^\top \Sigma$ is a graph Laplacian. It can easily be inverted with standard techniques as, for example, algebraic multigrid preconditioners. The diagonal preconditioner \mathbf{D}_Σ should not be obtained from $\mathbf{S}^\top \mathbf{Z} \mathbf{S}$ because of the large computational effort but from $\mathbf{H}_\Sigma^\top \mathbf{V}_{\text{near}} \mathbf{H}_\Sigma$.

When it comes to the vector potential in form of $\mathbf{Z}_{\Lambda\Lambda}$, the situation is different. In this case, there isn't a hierarchical basis that can be constructed with linear Lagrangian functions on unstructured meshes. In particular, the generalized Haar basis constructed on the dual mesh, cannot be applied for the direct regularization of the hypersingular operator! Instead, we must first transform the vector potential into a single layer potential. Consider that the single layer potential \mathcal{V} maps from $H^{-1/2}$ to $H^{1/2}$, while \mathcal{W} maps from $H^{1/2}$ to $H^{-1/2}$, where H^x are Sobolev spaces defined on the surface $\tilde{\Gamma}$, while for the hypersingular operator we have $\mathcal{W} : H^{1/2} \rightarrow H^{-1/2}$. The mapping properties of these two operators reflect that the single layer potential is an operator of order -1 (i.e., it is an integrative operator), while the hypersingular operator is an operator of order $+1$ (i.e., it is a derivative operator).

Our task is to find an operator \mathcal{X} such that $\mathcal{X}^\dagger \circ \mathcal{V} \circ \mathcal{X} : H^{-1/2} \rightarrow H^{1/2}$. Then $\mathcal{X}^\dagger \circ \mathcal{V} \circ \mathcal{X}$ is equivalent to the single layer potential (up to a compact operator), and we can use a generalized Haar basis as preconditioner [6]. The following proposition provides us with a correct \mathcal{X} .

Lemma 1. *Let \mathcal{W} be the hypersingular operator. Then the operator*

$$\mathcal{W} \circ \Delta_{\tilde{\Gamma}}^{-1} \circ \mathcal{W} \circ \Delta_{\tilde{\Gamma}}^{-1} \circ \mathcal{W} \quad (8)$$

maps from $H^{-1/2}/\mathbb{R}$ continuously into $H^{1/2}$ with $\Delta_{\tilde{\Gamma}}$ being the Laplace-Beltrami operator.

Proof. The hypersingular operator maps continuously from H^s/\mathbb{R} to H^{s-1} while the inverse Laplace-Beltrami operator maps continuously from H^{s-2}/\mathbb{R} to H^s . As the composition of a continuous mapping is continuous, the overall composition is a continuous mapping from $H^{-1/2}/\mathbb{R}$ to $H^{1/2}$. \square

The discretization of $\Delta_{\tilde{\Gamma}}$ is given by $\Lambda^\top \mathbf{G} \Lambda$, where $[\mathbf{G}]_{ij} = (\mathbf{f}_i, \mathbf{f}_j)_{L^2}$ is the Gram matrix of RWG functions. Again, such a matrix can rapidly be inverted with standard techniques like algebraic multigrid methods. As the Laplace-Beltrami operator and the hypersingular operator are discretized with linear Lagrangian functions, no Gram matrix appears between the *inverse* Laplace-Beltrami operator and the hypersingular operator. So we obtain as the discretization of (8)

$$\Lambda^\top \mathbf{Z} \Lambda (\Lambda^\top \mathbf{G} \Lambda)^{-1} \Lambda^\top \mathbf{Z} \Lambda (\Lambda^\top \mathbf{G} \Lambda)^{-1} \Lambda^\top \mathbf{Z} \Lambda. \quad (9)$$

The generalized Haar basis transformation matrix \mathbf{H}_Λ is constructed for a single layer potential that is discretized with the dual Buffa-Christiansen (BC) patch basis functions (for the definition of these functions, see [7]). The discretized operator of (9) is, however, equivalent to a single layer potential discretized with linear Lagrangian functions. In order to be able to apply \mathbf{H}_Λ to (9), we must link it with the mixed Gram matrix $[\mathbf{G}_{\text{pBC}\lambda}]_{ij} = (p_i^{\text{BC}}, \lambda_j)$ allowing for a change of basis. Modifying (9) accordingly leads to $(\mathbf{G}_{\text{pBC}\lambda}^{-1})^\top \mathbf{Z} \Lambda (\Lambda^\top \mathbf{G} \Lambda)^{-1} \Lambda^\top \mathbf{Z} \Lambda (\Lambda^\top \mathbf{G} \Lambda)^{-1} \Lambda^\top \mathbf{Z} \Lambda \mathbf{G}_{\text{pBC}\lambda}^{-1}$. Hence, we use as hierarchical basis for the vector potential $\mathbf{L} = \Lambda (\Lambda^\top \mathbf{G} \Lambda)^{-1} \Lambda^\top \mathbf{Z} \Lambda \mathbf{G}_{\text{pBC}\lambda}^{-1} \mathbf{H}_\Lambda$, and the overall hierarchical basis is $\mathbf{T} = [\mathbf{L} \quad \mathbf{S}]$.

4 Numerical Results

We verified the presented theory with two examples. First, a plane wave impinges on a cube with side length $\lambda/300$ with λ being the wavelength. Fig. 1a shows the largest singular value of $\Lambda^T Z \Lambda$ and $D_\Lambda L^T Z L D_\Lambda$ as function of the spectral index $1/h$, where h is the average edge length. We can see for $\Lambda^T Z \Lambda$ (i.e., when normalized loop functions are used), that the largest singular value of the vector potential grows linearly (the second smallest remains constant), while in the case of the proposed hierarchical basis, the grows largest singular value grows only logarithmically (the smallest singular value remains constant). The second example is a shuttle with a plane wave impinging on it. The shuttle has a length of $\lambda/2$ with the frequency being 1 MHz. In Fig. 1b, we can see the residual as a function of the number of iterations for the proposed preconditioner compared with a standard loop-star preconditioner. Clearly, the proposed preconditioner is advantageous.

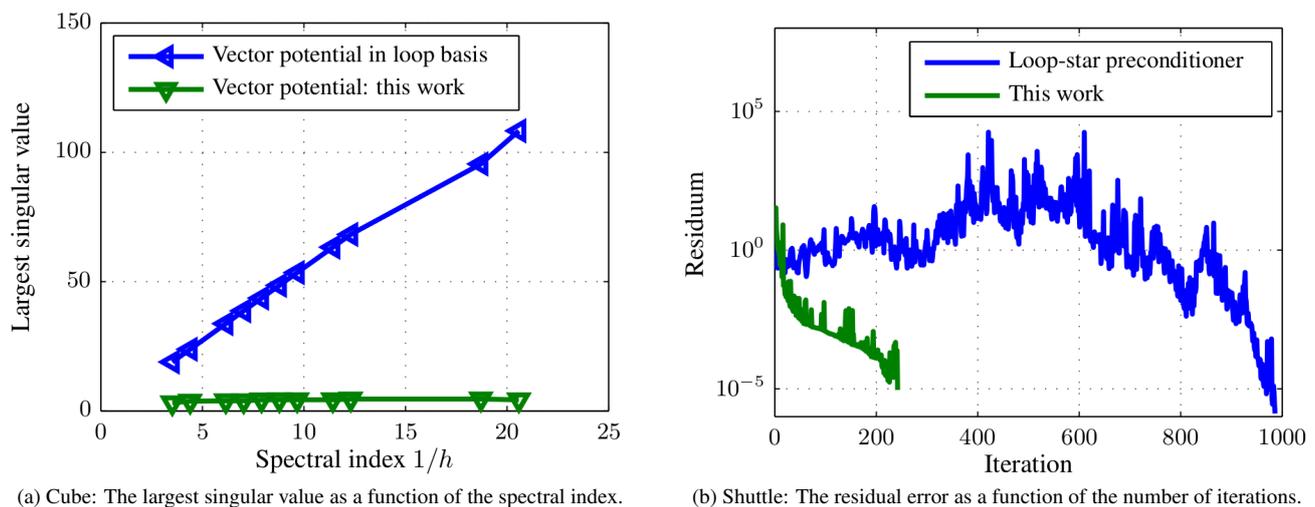


Figure 1: Numerical results for the cube and the shuttle. For both objects the frequency is 1 MHz.

5 Conclusion

In this paper, we presented a theoretical investigation of a hierarchical preconditioner for the EFIE applicable on structured and unstructured meshes. We have discussed the necessity to regularize both the vector and the scalar potential of the EFIE. Numerical results illustrated this necessity. In addition, the numerical results proved the effectiveness of the new preconditioner.

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