Sum Rules for Metamaterials in Parallel Plate Waveguides

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Abstract

An optical theorem and forward scattering approach for a bounded object confined between the plates of a parallel plate waveguide is investigated. A forward scattering sum rule together with the properties of positive real functions is used to show that the total absorption and scattering of an object is bounded by its static polarizability. A parallel plate waveguide and a parallel plate capacitor are used to estimate the dynamic and static properties, respectively. The results verify that the total cross section of an object integrated over the entire spectrum is proportional to the static polarizability.

1. Introduction

Metamaterials or artificially fabricated materials were first introduced by V.G. Veselago in 1967 and in 2000, Smith [1] published the first paper that demonstrated the existence of an artificial material with simultaneously negative parameters. The scattering properties of metamaterials and particularly frequency selective structures have been a popular research topic for more than a decade. The forward scattering in electromagnetic theory is a classical problem that together with the optical theorem relates the scattering in all directions to the scattering in the forward direction. The total cross-section from a scatterer is proportional to the real part of the forward scattering amplitude. The forward scattering can be used to construct a positive real (PR) function [2]. The forward scattering sum rule using properties of positive real functions relates the total cross section (absorption and scattering in all directions) to the static polarizability of the object [3, 4].

A relation between the scattered far-field amplitude and the total cross section for a scatterer bounded between the plates of a parallel plate waveguide is derived in Section 2. The related sum rule for parallel plate waveguides is defined in Section 3. The forward scattering of objects can be determined using a parallel plate waveguide and the static polarizability using a parallel plate capacitor as discussed in Section 4. In Section 5 we show that the total (extinction) cross section of a resonant structure over the entire spectrum is proportional to the static polarizability of the object. The measured values are verified by numerical simulations. The paper is concluded in Section 6.

2. Bounded scatterer in a parallel plate waveguide

In this section we derive an optical theorem for an object which is confined between the plates of a parallel plate waveguide. As shown in figure 1, an imaginary cylindrical surface S with a normal unit vector $\hat{\boldsymbol{n}} = \hat{\boldsymbol{\rho}} = \hat{\boldsymbol{x}}\sin\phi + \hat{\boldsymbol{y}}\cos\phi$, circumscribes the object. An arbitrarily shaped object is excited by a linearly polarized planewave propagating in the x-direction $\boldsymbol{E}_{\mathrm{i}} = \hat{\boldsymbol{z}}E_0\mathrm{e}^{-\mathrm{j}kx}$, where k is the wavenumber, j denotes the imaginary unit, and the related time convention is $\mathrm{e}^{\mathrm{j}\omega t}$. The sum of the absorbed and scattered power is the total power, $P_{\mathrm{tot}} = P_{\mathrm{a}} + P_{\mathrm{s}}$, that can be expressed as [5, p. 501]

$$P_{\mathrm{tot}} = \frac{1}{2}\,\mathrm{Re}\int_S \hat{\boldsymbol{n}}\cdot\big(-(\boldsymbol{E}_{\mathrm{i}}+\boldsymbol{E}_{\mathrm{s}})\times(\boldsymbol{H}_{\mathrm{i}}+\boldsymbol{H}_{\mathrm{s}})^* + \boldsymbol{E}_{\mathrm{s}}\times\boldsymbol{H}_{\mathrm{s}}^*\big)\,\mathrm{dS} = -\frac{1}{2}\,\mathrm{Re}\int_S \hat{\boldsymbol{n}}\cdot(\boldsymbol{E}_{\mathrm{s}}\times\boldsymbol{H}_{\mathrm{i}}^* + \boldsymbol{E}_{\mathrm{i}}\times\boldsymbol{H}_{\mathrm{s}}^*)\,\mathrm{dS}, \quad (1)$$

where $\boldsymbol{E}_{\mathrm{s}}$ denotes the scattered field, $\boldsymbol{E}_{\mathrm{s}} = \boldsymbol{E} - \boldsymbol{E}_{\mathrm{i}}$, defined as the difference between the total field, $\boldsymbol{E}$, and the incident field, $\boldsymbol{E}_{\mathrm{i}}$. By inserting the incident field and using $\hat{\boldsymbol{y}} = \hat{\boldsymbol{z}}\times\hat{\boldsymbol{x}}$ we get

$$P_{\mathrm{tot}} = -\frac{1}{2}\,\mathrm{Re}\int_S E_0^*\mathrm{e}^{\mathrm{j}kx}\hat{\boldsymbol{n}}\cdot(-\boldsymbol{E}_{\mathrm{s}}\times\hat{\boldsymbol{y}}\eta_0^{-1} + \hat{\boldsymbol{z}}\times\boldsymbol{H}_{\mathrm{s}})\,\mathrm{dS} = \frac{-b}{2\eta_0}\,\mathrm{Re}\int_0^{2\pi} E_0^*\mathrm{e}^{\mathrm{j}kx}\hat{\boldsymbol{z}}\cdot\langle\boldsymbol{E}_{\mathrm{s}}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{x}}) + \eta_0\boldsymbol{H}_{\mathrm{s}}\times\hat{\boldsymbol{n}}\rangle\,\mathrm{d}\phi \quad (2)$$

where $\langle . \rangle$ is a short hand for the mean value of the fields over the height of the waveguide and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ denotes the free space wave impedance. The scattered fields inside the parallel plate waveguide are decomposed into TE and TM modes:

$$\text{TE modes}: \begin{cases} \boldsymbol{E}_{\mathrm{s}} = E_\rho\hat{\boldsymbol{\rho}} + E_\phi\hat{\boldsymbol{\phi}} \\ \boldsymbol{H}_{\mathrm{s}} = H_\rho\hat{\boldsymbol{\rho}} + H_\phi\hat{\boldsymbol{\phi}} + H_z\hat{\boldsymbol{z}} \end{cases} \quad \text{and TM modes}: \begin{cases} \boldsymbol{E}_{\mathrm{s}} = E_\rho\hat{\boldsymbol{\rho}} + E_\phi\hat{\boldsymbol{\phi}} + E_z\hat{\boldsymbol{z}} \\ \boldsymbol{H}_{\mathrm{s}} = H_\rho\hat{\boldsymbol{\rho}} + H_\phi\hat{\boldsymbol{\phi}} \end{cases}$$

According to (2) and figure 1, $\hat{\boldsymbol{z}}\cdot\boldsymbol{E}_{\mathrm{s}} = E_z^{\mathrm{TM}}$ and $\boldsymbol{H}_{\mathrm{s}}\times\hat{\boldsymbol{n}} = -H_\phi\hat{\boldsymbol{z}}$ for both TE and TM modes we get

$$P_{\mathrm{tot}} = \frac{-1}{2\eta_0}\,\mathrm{Re}\big\{E_0^*\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_0^b\int_0^{2\pi}\mathrm{e}^{\mathrm{j}kx}\big(E_z^{\mathrm{TM}}\cos\phi - \eta_0(H_\phi^{\mathrm{TE}} + H_\phi^{\mathrm{TM}})\big)\rho\,\mathrm{d}\phi\,\mathrm{d}z\big\}, \quad (3)$$

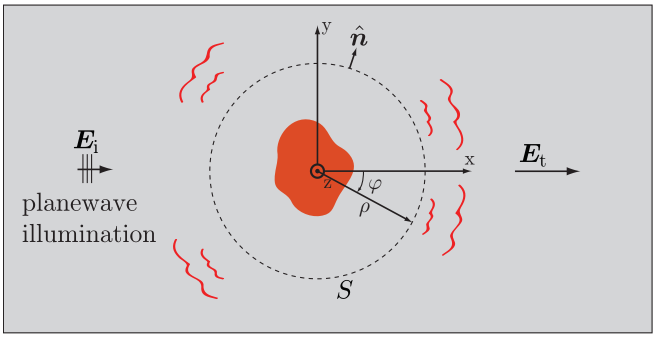

Figure 1: Top view of the parallel plate waveguide. The imaginary surface S is a cylinder of radius ρ and normal vector $\hat{n}$, that encloses the object.

where the integral is on a circular cylindrical surface defined by $\sqrt{x^2+y^2}=\rho=$ constant. The expansion of the fields in cylindrical waves for the mn mode is given by [6]

$$E^{\mathrm{TM}}_{z,mn}(\rho,\phi,z) = -\mathrm{j}B_{mn}\frac{k_\rho^2}{\omega\mu_0\epsilon_0}H^{(2)}_m(k_\rho\rho)\cos(m\phi)\cos(\frac{n\pi}{b}z) \tag{4}$$

$$H^{\mathrm{TM}}_{\phi,mn}(\rho,\phi,z) = -B_{mn}\frac{k_\rho}{\mu_0}H^{(2)\prime}_m(k_\rho\rho)\cos(m\phi)\cos(\frac{n\pi}{b}z) \tag{5}$$

$$H^{\mathrm{TE}}_{\phi,mn}(\rho,\phi,z) = \mathrm{j}A_{mn}\frac{mn\pi}{\omega\mu_0\epsilon_0\rho b}H^{(2)}_m(k_\rho\rho)\sin(m\phi)\cos(\frac{n\pi}{b}z), \tag{6}$$

where μ_0 and ϵ_0 denote the free space permeability and permittivity, respectively. $m=(0,1,\ldots)$ is the index for the cylindrical wave and n for the waveguide modes, A_{mn} and B_{mn} are the frequency dependent expansion coefficients for TE (n=1,2,...) and TM (n=0,1,...) modes, respectively and $k_\rho=\sqrt{k^2-(n\pi/b)}$ is the wavenumber in the ρ direction. Vertical variations are solely based on $\cos(n\pi z/b)$ and the integration over the z-axis is $c_n=\int_0^b\cos(n\pi z/b)\,\mathrm{d}z=b$ for $n=0$, and $=0$ for $n>0$. It should be pointed out that all higher modes have zero contribution and only the zeroth order mode remains and thus, $k_\rho=k$. The $\phi-$dependence in (3) is in the exponential coefficient and also in $\cos(m\phi)$ which is hidden in the fields. By using $x=\rho\cos(\phi)$ and the integral representation of the Bessel function [5, p. 140] we get $\int_0^{2\pi}\mathrm{e}^{\mathrm{j}k\rho\cos\phi}\cos(m\phi)\,\mathrm{d}\phi=2\pi\mathrm{j}^mJ_m(k\rho)$, where $J_m(k\rho)$ is the Bessel function of the first kind and order m. Using the fact that only the zeroth order mode contributes in the optical theorem, the total power in (3) is simplified to

$$P_{\mathrm{tot}}=\frac{\rho}{2\eta_0}\,\mathrm{Re}\,\big\{E_0^*\sum_{m=0}^{\infty}2\pi\mathrm{j}^mkc_0B_{m0}b\big(H^{(2)}_m(k\rho)J'_m(k\rho)-H^{(2)\prime}_m(k\rho)J_m(k\rho)\big)\big\}, \tag{7}$$

where c_0 is the speed of light in vacuum. The statement inside the bracket is similar to the Wronskian of the Bessel functions. By expanding the Hankel functions to Bessel and Neumann functions and using the Wronskian [7], *i.e.*, $\big(H^{(2)}_mJ'_m(k\rho)-H^{(2)\prime}_m(k\rho)J_m(k\rho)\big)=\frac{2\mathrm{j}}{\pi k\rho}$ the total power can be simplified to

$$P_{\mathrm{tot}}=\frac{1}{2\eta_0}\sum_{m=0}^{\infty}\mathrm{Re}\,\big\{\frac{4}{k}E_0^*\mathrm{j}^{m+1}kc_0bB_{m0}\big\}. \tag{8}$$

This is valid for any ρ and by normalizing the total power with the incident power flux, $|E_0|^2/2\eta_0$, the total cross section using the definition of the far-field amplitude, f_s, in two dimensions [8] can be written as

$$\sigma_{\mathrm{tot}}=-4b\,\mathrm{Re}\left\{\frac{f_\mathrm{s}(k)}{\mathrm{j}kE_0}\right\},\qquad\text{where } f_\mathrm{s}(k)=\frac{\mathrm{j}k}{4}\int_0^{2\pi}\mathrm{e}^{\mathrm{j}kx}\hat{\boldsymbol{z}}\cdot\langle\boldsymbol{E}_\mathrm{s}(\hat{\boldsymbol{n}}\cdot\hat{\boldsymbol{x}})+\eta_0\boldsymbol{H}_\mathrm{s}\times\hat{\boldsymbol{n}}\rangle\,\mathrm{d}\phi. \tag{9}$$

3. Forward scattering sum rules for parallel plate waveguides

The total cross section of any scatterer is bounded by the static polarizability of the object using the forward scattering sum rule [2, 4]. The forward scattering sum rule is derived using positive real function, $\mathscr{P}(\kappa)$, generated by (9), *i.e.*, $\mathscr{P}(\kappa)=-\frac{4b}{\kappa E_0}f_\mathrm{s}(\kappa)$, where $\sigma_{\mathrm{tot}}=\mathrm{Re}\{\mathscr{P}(\kappa)\}$ and $\kappa=\zeta+\mathrm{j}k$. Using the expansions in the fields reported in [9], at low frequencies the forward scattering weighted by the frequency is proportional to the static polarizability, *i.e.*, $\mathscr{P}(\kappa)/\kappa=\gamma$ as $\kappa\hat{\to}0$, where $\hat{\to}$ denotes the limit inside the right half plane and γ is the static polarizability. The forward scattering sum rule that relates all spectrum interaction between the electromagnetic fields and the object to the static polarizability and the general approach is [4]

$$\frac{2}{\pi}\int_0^\infty\frac{\sigma_{\mathrm{tot}}(k)}{k^2}\,\mathrm{d}k=\gamma. \tag{10}$$

The left hand side is an integration of the total cross section, σ_{tot}, over all wavenumbers (all frequencies) and the right hand side is the static polarizability.

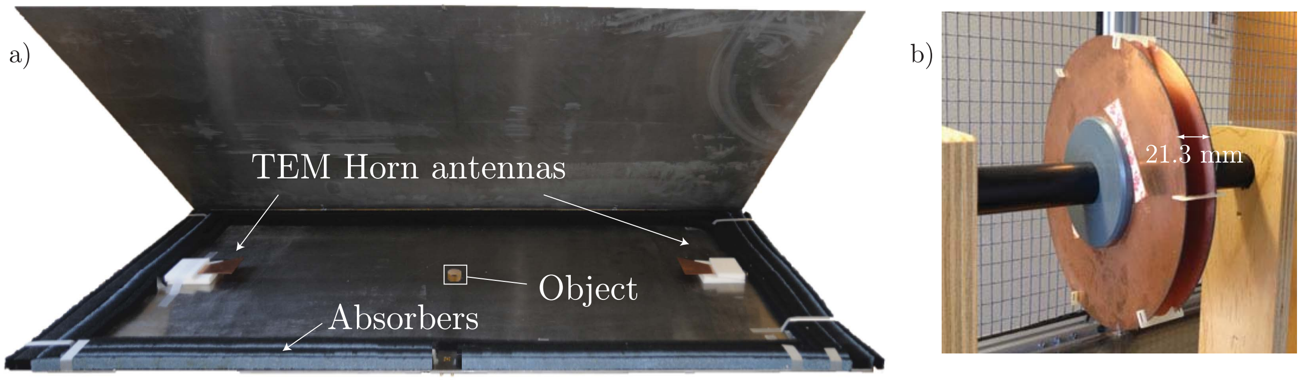

Figure 2: a) Parallel plate waveguide consisting of two TEM horn antennas and absorbers. b) parallel plate capacitor. The distance between the plates in both cases is 21.3 mm.

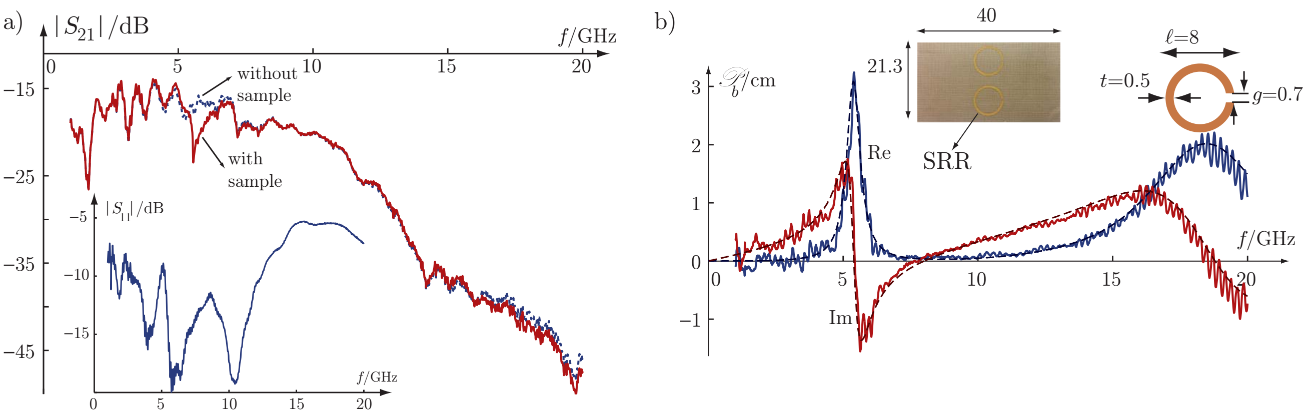

Figure 3: a) Measured values of $|S_{21}|$ in the presence (solid) and the absence (dashed) of the object. The lower figure shows the $|S_{11}|$ in the absence of the object. b) Measured values (solid curves) and the simulated values (dashed curves) for the complex forward scattering of a resonant structure consisting of two split ring resonators (SRR).

4. Methodology

We use a parallel plate waveguide in order to measure the dynamic behavior of a scatterer and a parallel plate capacitor to measure the static polarizability in the right hand side of (10). The polarizability is infinite for objects that short circuit the plates. The integration on the left hand side of the sum rule is also infinite for this case making (10) less useful.

A parallel plate waveguide consisting of two wideband TEM horn antennas, as shown in figure 2a is used to determine the scattering properties of different materials over the frequency range $[2-20]$ GHz. The antennas are separated by 98 cm facing each other and the object is placed at the midpoint between two antennas. The reflection coefficient, $|S_{11}|$ and the transmission inside the waveguide $|S_{21}|$ are shown in figure 3a. The forward scattering inside the waveguide is found by measuring S_{21} in the presence ($S_{21,\text{obj}}$) and the absence ($S_{21,\text{emp}}$) of the object and is given by [2, 4]

$$\mathscr{P}_b(\mathrm{j}k) = \frac{-4f_\mathrm{s}}{\mathrm{j}kE_0} = \frac{4}{\mathrm{j}k}\left(1 - \frac{S_{21,\text{obj}}}{S_{21,\text{emp}}}\right)\sqrt{\frac{\pi \mathrm{j}kd_1(d-d_1)}{2d}}, \tag{11}$$

A parallel plate capacitor, as shown in figure 2b, is used to determine the static polarizability. The capacitor consists of two copper plates supported by a plastic holder and is placed in the middle of a Faraday cage and a ground plane to reduce the interferences. The polarizability is found by inserting the object inside the capacitor and measuring the capacitance change between the absence and the presence of the object [10], *i.e.*, $\gamma = \Delta C d^2/\epsilon_0$, where $\Delta C = C_\text{obj} - C_\text{emp}$ is the difference between the capacitance in presence, C_obj, and the absence, C_emp, of the object, d is the separation between the two plates and ϵ_0 is the vacuum permittivity.

5. Example and results

The forward scattering of a resonant structure consisting of two split ring resonators (SRR) printed on 127 μm thick Arlon Diclad880 with relative permittivity $\epsilon_r = 2.17$ is shown in figure 3b. The measurement is performed in a parallel plate waveguide and the solid lines are the real and imaginary parts of the measured forward scattering, $\mathscr{P}_b$, and the dashed lines represent the simulated values. The simulations are carried out in the FDTD solver in CST microwave studio with periodic boundary conditions applied to the top and the bottom of the object. It should be noted that the periodic boundary condition approximates the parallel plate waveguide in this particular example. The measured values are in good agreement with the simulations and the advantage of simulating the object is to estimate the signal without noise and also verify the measurement setup. The integration on the left hand side of the sum rule (10) for the measured values gives $0.78\,\mathrm{cm}^3$ whereas simulated value is $0.87\,\mathrm{cm}^3$. The measurement is noisy particularly at lower frequencies and causes negative values of the forward scattering that decreases the value of the integral on the left hand side. One way to mitigate the noise level is to use the convex optimization method as discussed in [2] for passive measurement systems.

The static polarizability on the right hand side of the sum rule is measured using the parallel plate capacitor (figure 2b) by estimating the changes in the capacitance (ΔC) and also simulated using the finite element solver from Comsol multiphysics. The measured polarizability is estimated to $\gamma = 0.97\,\mathrm{cm}^3$, whereas the simulated one is $\gamma = 0.93\,\mathrm{cm}^3$. The results show that the static polarizability of the object impose an upper bound on the total cross section of the object. The measured total cross section, σ_{tot}, of the object with this experimental approach is about 80% of the static polarizability.

6. Conclusions

An experimental approach to the forward scattering sum rule for parallel plate waveguides is investigated. The optical theorem is used to relate the total cross section to the forward scattering for an object bounded by an imaginary cylinder inside the waveguide and the related sum rule is used to find the total cross section from the static polarizability of the object. The theoretical approach is followed by measurements of the forward scattering and the static polarizability and are verified by numerical simulations. The results show a good agreement between the simulations and the measurements and also verifies that the total interaction between the electric field and an object is bounded by the static polarizability that is measured in a parallel plate capacitor.

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