

The Most General Classes of Tellegen Media Reducible to Simple Reciprocal Media: a Geometrical Approach

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Abstract

Duality mappings allow to transform a nonreciprocal achiral bi-isotropic medium (Tellegen medium) into a conventional reciprocal material leaving the free space invariant. In particular, the analytical solutions of electromagnetic problems involving a single Tellegen medium and a vacuum can be found by applying the inverse duality transformation to the solution of the duality transformed problem wherein the materials are conventional reciprocal media. Here, based on the geometrical interpretation of duality transformations in the Riemann sphere, we derive the most general classes of Tellegen media that are reducible to simple isotropic media (SIMs) under the same duality transformation. It is shown that Tellegen media can be identified with the points of the Riemann sphere. Moreover, duality transformations are classified into different categories according to their geometrical actions on the sphere. We apply the developed theory to periodic structures formed by Tellegen media showing how the wave propagation in these complex structures can be easily studied using duality transformations. Furthermore, to unveil the role of the nonreciprocal response, we investigate the wave propagation in Tellegen periodic structures wherein the pertinent materials cannot be reduced to conventional media.

1. Introduction

Duality transformations were originally defined as linear transformations of the electromagnetic field vectors [1]. These transformations have the ability to take a nonreciprocal achiral bi-isotropic medium (Tellegen medium) [2] to a simple isotropic medium leaving the vacuum invariant [1, 3, 4]. Duality transformations are also useful in electromagnetic problems involving known boundaries such as perfect electric conductors (PEC), perfect magnetic conductors (PMC) and perfect electromagnetic conductors (PEMC) [5]. The duality transformed fields satisfy the Maxwell's equations in a medium with transformed constitutive parameters.

Some authors claimed some time ago that Tellegen media cannot exist in nature because they violate the Post constraint [1]. However, other researchers have experimentally demonstrated the realization of Tellegen artificial particles [6]. In addition, there is experimental evidence that the natural composite chromium sesquioxide, Cr₂O₃ [7], violates the Post constraint. More recently, it was suggested that topological insulators can have in some circumstances an electromagnetic behavior consistent with that of Tellegen (axion) media [8].

In this work, we give a geometrical interpretation for the duality transformations applied to Tellegen media. Furthermore, it is proven that isorefractive Tellegen media are isomorphic to the Riemann sphere [9]. Based on a geometrical interpretation, duality transformations are classified as generalized rotations, Lorentz boosts or Galilean boosts. The developed geometrical concepts allow us to derive the most general classes of Tellegen media that are simultaneously reducible to SIM by a suitable duality transformation. As an example of the application of this result, we consider the wave propagation problem in periodic structures formed by Tellegen media. It is shown that periodic structures formed by two Tellegen media can always be reduced to periodic structures formed by simple isotropic materials. In addition, we also consider Tellegen periodic structures wherein the relevant media cannot be reduced to simpler media by a duality transformation. It is shown that the corresponding band diagrams are not topologically equivalent to those of conventional dielectrics, unveiling in this manner the signature of the nonreciprocal material response.

2. Duality Transformations for Tellegen Media in the Riemann Sphere

Tellegen media are achiral nonreciprocal bi-isotropic media [2]. In this work, it is convenient to write the constitutive relations of a lossless Tellegen medium in a matrix form as

$$\begin{pmatrix} \mathbf{D} / \varepsilon_0 \\ \mathbf{B} \mathbf{c} \end{pmatrix} = \bar{\mathbf{M}} \cdot \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}, \quad \text{with} \quad \bar{\mathbf{M}} = \begin{pmatrix} \varepsilon & \kappa \\ \kappa & \mu \end{pmatrix} \quad (1)$$

where $\bar{\mathbf{M}}$ is the material matrix of a Tellegen medium, η_0 is the vacuum wave impedance and $n = \sqrt{\varepsilon\mu - \kappa^2}$ is the refractive index. On the other hand, the constitutive parameters and the wave impedance are parameterized as

$$\varepsilon = |n|(\Gamma^2 + M^2)/\Gamma, \quad \mu = |n|/\Gamma, \quad \kappa = |n|M/\Gamma, \quad \eta = \sqrt{1/(M^2 + \Gamma^2)}, \quad (2)$$

where the admittance $M = \kappa/\mu$ is associated with the magnetoelectric coupling whereas the admittance $\Gamma = |n|/\mu$ gives the principal part of (1). From (2) it is clear that any family of isorefractive Tellegen media, with a constant $|n|$, can be represented in the (M, Γ) plane. Thus, using the stereographic projection \hat{P}_r [9] it is possible to map each point in the complex plane $w = M + i\Gamma$ into a point $p(x, y, z)$ on the Riemann sphere [Fig. 1a]. Some particular cases of Tellegen media with special interest correspond to specific limits of the admittances M and Γ . For example, the condition $M = 0$ yields the SIMs, and the condition $\Gamma = 0$ corresponds to the PEMCs. Thus, the SIMs and the PEMCs are represented by lines in the complex plane, and by meridians in the Riemann sphere as illustrated in Fig. 1a. The PEC and the PMC cases defined by $\eta = 0$ and $\eta \rightarrow \infty$, respectively, are the north and south poles of the unit sphere. In addition, the points on the hemisphere $y > 0$ correspond to Tellegen media with positive ε and μ parameters, whereas the region $y < 0$ is associated with media with negative values of ε and μ , i.e. with generalized nonreciprocal Veselago media with negative refractive index [10].

Duality transformations [1, 3] are defined as linear mappings of the electromagnetic fields of the form:

$$\begin{pmatrix} \mathbf{E}_d \\ \eta_0 \mathbf{H}_d \end{pmatrix} = \bar{\mathbf{S}} \cdot \begin{pmatrix} \mathbf{E} \\ \eta_0 \mathbf{H} \end{pmatrix}, \quad \text{with } \bar{\mathbf{S}} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad (3)$$

where $\bar{\mathbf{S}}$ is a 2×2 real-valued matrix with constant elements (independent of the spatial coordinates) and s_{11} , s_{12} , s_{21} and s_{22} are real-valued parameters. It is well-known that the duality transformed fields \mathbf{E}_d and \mathbf{H}_d are solutions of the Maxwell's equations in a transformed structure described by the transformed material matrix, $\bar{\mathbf{M}}_d = \det(\bar{\mathbf{S}})(\bar{\mathbf{S}}^{-1})^T \cdot \bar{\mathbf{M}} \cdot \bar{\mathbf{S}}^{-1}$. From this formula follows that $\bar{\mathbf{M}} \rightarrow \bar{\mathbf{M}}_d$ induces a mapping of the Riemann sphere into itself. A detailed analysis shows that a duality transformation always has two fixed points. The fixed points correspond to the media that are unchanged by the transformation. There are three possibilities for the location of the fixed points in the Riemann sphere: *i*) the fixed points are two Tellegen media located in the hemispheres $y > 0$ and $y < 0$, and that are the mirror of one another with the respect to the $y=0$ plane; *ii*) the fixed points lie in the PEMC circle ($x^2 + z^2 = 1$ meridian) and are not coincident; *iii*) the fixed points are coincident and lie the PEMC circle. Depending on the case (i), (ii) or (iii) described above, we classify a duality transformation as a *generalized rotation*, a *generalized Lorentz boost* or a *generalized Galilean boost*, respectively. The action of a generalized rotation is to rotate the points of the Riemann sphere around a fixed point. On the other hand, the geometrical action of a Lorentz boost is to drag the points in the Riemann sphere from one of the fixed points (the source) to the other fixed point (the sink). The Galilean boost is analogous to the Lorentz boost, except that the sink and the source are coincident.

A detailed analysis shows that the duality transformations with $\det \bar{\mathbf{S}} = 1$ can be written in the exponential form as $\bar{\mathbf{S}} = e^{u \sigma \hat{\mathbf{n}}}$, where u is some real-valued parameter, $\hat{\mathbf{n}} = (n_x, n_y, n_z)$ is a vector, and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ represent the Pauli matrices. Moreover, it can be shown that for the generalized rotations $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = -1$, for the Lorentz boosts $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 1$, whereas the Galilean boosts have $\hat{\mathbf{n}} \cdot \hat{\mathbf{n}} = 0$. Here, n_x, n_z are real-valued parameters and n_y is imaginary pure.

Using these ideas, it is possible to derive the most general sets of Tellegen media that are reducible to the SIM family under the same duality transformation. Each of these sets is named a *Tellegen class*. It is possible to prove that duality transformations are conformal mappings. Thus, the SIM circle [see Fig. 1a] is taken by any duality mapping into another circle. Moreover, the duality transformed SIM circle is such that if a point lies in the circle then its "mirror image" point with respect to the $y=0$ plane also belongs to the circle. This demonstrates that the most general *Tellegen classes* that can be mapped into the SIM circle are represented in the Riemann sphere by circles invariant to reflections with respect to the $y=0$ plane. This is one of the key results of the article.

Any two Tellegen media, (Tel_1) and (Tel_2) , in the hemisphere $y > 0$ determine a Tellegen class. Thus, they can be mapped by some duality transformed into two SIM, (SIM_1) and (SIM_2) , in the hemisphere $y > 0$. This can be visualized with the composition of two generalized rotations, $\bar{\mathbf{S}}_{R,1}$ and $\bar{\mathbf{S}}_{R,2}$. The first transformation $\bar{\mathbf{S}}_{R,1} = e^{(\theta_1/2) \sigma \hat{\mathbf{n}}}$ takes (Tel_1) defined by the constitutive parameters $(\varepsilon_1, \mu_1, \kappa_1)$ to (SIM_1) with a rotation angle $\theta_1 = \arctan[2\kappa_1/(\mu_1 - \varepsilon_1)]$, as outlined in Fig. 1ci. On the other hand, it transforms (Tel_2) to another Tellegen media (Tel'_2) leaving the vacuum invariant (fixed point). The second transformation $\bar{\mathbf{S}}_{R,2} = e^{(\theta_2/2) \sigma \hat{\mathbf{n}}}$ (see Fig. 1cii) must fix the point (SIM_1) defined by (ε'_1, μ'_1) and transforms (Tel'_2) to (SIM_2) using the angle $\theta_2 = \arctan[2\kappa'_2/(\mu'_2/\eta'_1 - \varepsilon'_2 \eta'_1)]$ where $(\varepsilon'_2, \mu'_2, \kappa'_2)$ are the constitutive parameters of (Tel'_2) and $\eta'_1 = \sqrt{\mu'_1/\varepsilon'_1}$ is the wave impedance of the medium (SIM_1) . The medium (SIM_2) is characterized by the constitutive parameters (ε'_2, μ'_2) .

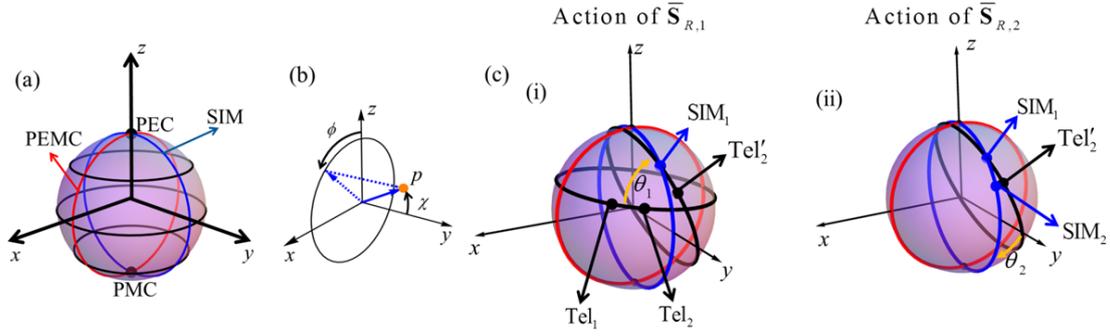


Fig. 1 (a) Geometrical representation of Tellegen media. Some particular cases are shown: the PEC and PMC poles and the PEMC and SIM meridians. (b) The spherical coordinates of a point p on the surface of the Riemann sphere. (c) Transformation of Tellegen media into SIMs. The media (Tel_1) and (Tel_2) are characterized by the spherical coordinates $(\chi_1 = \pi/4, \phi_1 = \pi/3)$ and $(\chi_2 = \pi/7, \phi_2 = \pi/4)$, respectively. (i) Tellegen media represented by (Tel_1) and (Tel_2) are transformed into (SIM_1) and (Tel'_2) under the transformation $\bar{S}_{R,1}$. (ii) The Tellegen medium represented by (Tel'_2) is taken to (SIM_2) by the transformation $\bar{S}_{R,2}$, leaving (SIM_1) invariant.

3. Photonic Crystals with Tellegen media

Next, we characterize the dispersion of the Bloch modes of photonic crystals formed by Tellegen media. In Fig. 2ai, we consider the scenario wherein the crystal is formed by two different Tellegen materials. As previously discussed, in this case it is always possible to reduce the structure to a photonic crystal formed by reciprocal media. This can be done using a duality transformation $\bar{S}_{T \rightarrow S} = \bar{S}_{R,2} \cdot \bar{S}_{R,1}$, such that the Tellegen photonic crystal is transformed into a SIM photonic crystal. Thus, (Tel_1) and (Tel_2) , are transformed into (SIM_1) and (SIM_2) , leaving the thicknesses d_1 and d_2 and the z -propagation constants $\beta_1 = n_1 \omega / c$ and $\beta_2 = n_2 \omega / c$ invariant. Importantly, duality transformations only act over the electromagnetic fields, leaving the spatial coordinates $\mathbf{r} = (x, y, z)$ and the time coordinate t invariant. As consequence, the band structure of a photonic crystal is invariant under a duality mapping, i.e the dispersion diagrams ω vs. k_z are precisely the same for the original crystal and for a duality-transformed crystal. It is well known that the dispersion equation for the Bloch waves of the SIM photonic crystal is given by

$$\cos(k_z a) = \cos(\beta_1 d_1) \cos(\beta_2 d_2) - \frac{[\beta_1^2 \mu'_2 + \beta_2^2 \mu'_1]}{\beta_2 \beta_1 \mu'_1 \mu'_2} \frac{\sin(\beta_1 d_1) \sin(\beta_2 d_2)}{2}, \quad (4)$$

being k_z the propagation constant of the Bloch wave. Note that for propagation along the z -direction ($k_x = k_y = 0$) the Bloch waves are degenerate because the TE and TM waves have the same characteristic dispersion.

On the other hand, the band structure of the original Tellegen photonic crystal can be directly calculated using scattering matrices [11]. Indeed, the dispersion equation of a Tellegen periodic structure can be expressed as:

$$\det \left(\begin{bmatrix} e^{-ik_z a} \bar{\mathbf{T}}^L - \bar{\mathbf{I}} & \bar{\mathbf{R}}^R \\ \bar{\mathbf{R}}^L & e^{ik_z a} \bar{\mathbf{T}}^R - \bar{\mathbf{I}} \end{bmatrix} \right) = 0, \quad (5)$$

where $\bar{\mathbf{I}}$ is the 2x2 identity unitary matrix. In the above, the matrices $(\bar{\mathbf{R}}^L, \bar{\mathbf{T}}^L)$ and $(\bar{\mathbf{R}}^R, \bar{\mathbf{T}}^R)$ represent the reflection and transmission matrices for the transverse components of the electric field calculated for plane wave normal incidence on a single cell of the structure standing alone in a vacuum [11]. The superscripts ‘‘R’’ and ‘‘L’’ indicate if the incoming wave comes from the left (propagates along the $+z$ direction) or from the right hand side (propagates along the $-z$ direction).

In the Fig. 2aii we plot the band diagrams of the Tellegen photonic crystal obtained from the solutions of the equations (4) and (5). As expected, the results are exactly coincident. As previously mentioned, each line in the band structure is associated with two distinct modes because of polarization degeneracy. Thus, for Tellegen media in the same *class* the Bloch modes are doubly degenerated modes.

We also investigated the case where the Tellegen photonic crystal is irreducible to a SIM photonic crystal. It is evident that we need to consider crystals formed by at least three different materials [see Fig. 2bi]. As an example, we consider the combination of (Tel_1) , (Tel_2) and the vacuum, and choose the parameters of the Tellegen materials such that the three materials cannot be simultaneously reduced to SIM. The band diagram of this photonic crystal can be calculated again based on the dispersion equation (5). Notably, the numerical results show that when the Tellegen photonic crystal cannot be reduced to a SIM photonic crystal the Bloch modes are no longer degenerated, and thus we have two distinct dispersion curves, as shown in Fig. 2bii.

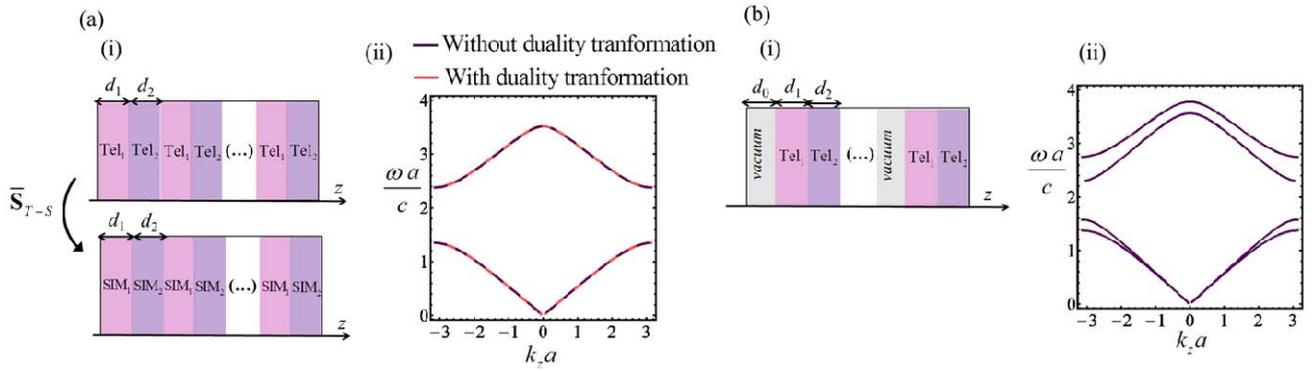


Fig. 2 Photonic crystals involving Tellegen media characterized by $(\epsilon_1 = 3, \mu_1 = 2, \kappa_1 = 1.5)$ and $(\epsilon_2 = 1, \mu_2 = 3, \kappa_2 = 1)$. (a) Periodic cell with media in the same class. (i) Geometry of the Tellegen crystal. (ii) Band diagrams with $d_1 = d_2 = 0.5$. (b) Periodic cell with media irreducible to a Tellegen class. (i) Geometry of the Tellegen crystal. (ii) Band diagrams with $d_1 = d_2 = 0.4$ and $d_0 = 0.2$.

4. Conclusion

We developed a geometrical interpretation of duality transformations for Tellegen media in the Riemann sphere. It was shown that each family of isorefractive Tellegen media can be identified with the unit sphere. A classification of duality transformations has surfaced naturally from the geometrical action of the transformations in the Riemann sphere. The fixed points of the transformations can be either Tellegen media or electromagnetic boundaries depending on the type of transformation. We have demonstrated that these geometric ideas lead to a clearer understanding of the opportunities created by duality transformations in the reduction of Tellegen media to simpler media. Specifically, we have shown that any two arbitrary Tellegen materials can always be reduced to SIMs under a composition of two generalized rotations. Moreover, the most general classes of Tellegen media reducible to SIMs were characterized. In particular, our theory was applied to the study of wave propagation in Tellegen photonic crystals, demonstrating how some complex structures can be easily handled with duality transformations. It was shown that the band structure of Tellegen photonic crystals stays invariant under a duality transformation. In particular, the band structure of Tellegen photonic crystals formed by media in the same class can be found from the band structure of conventional SIM photonic crystals. We also investigated the band structures of Tellegen crystals irreducible to SIMs, proving that they are topologically distinct from those of conventional media. This reveals the distinctive and unique signature of Tellegen materials in wave propagation problems. Finally, we would like to note that our theory can be applied to a wide range of propagation and radiation problems.

5. References

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