Analytic solution for the reflection of cylindrical wave at planar interfaces

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Abstract

In this paper, we present an analytical solution for the spectral integral representing the reflected cylindrical wave at a plane interface. The analytical solution involves the Bessel and Anger-Weber functions. We compare the analytical expression with the results obtained with a quadrature method previously proposed in the literature, finding very good agreement. Moreover, we find that the quadrature method is affected by some numerical errors in the near field, due to the highly oscillating integral's kernel. We try a physical interpretation of the result obtained connecting it to the well known Image Principle.

1. Introduction

The reflection of cylindrical waves at planar interfaces is an important topic, in both electromagnetic and acoustic researches. The typical application of this problem is to the scattering by cylindrical objects placed near to plane interfaces. These scattering problems are of great importance in many applications, to remote sensing of buried pipes, conduits, and cables, to the study of the interaction between buried objects and surrounding media, to the communication through the Earth, and to wave radiation through metamaterials. In these applications the scattered wave by the cylinder can be expressed as the superposition of Cylindrical Waves (CWs). The CWs are reflected by the plane interface and the reflection can be considered with the plane-wave spectrum of the CW, by multiplying each elementary wave by the Fresnel reflection coefficient [1]. The Reflected cylindrical Wave (RW) is expressed as a spectral integral. The numerical evaluation of the spectral integral is not a trivial task because of the high oscillations of the kernel, and it is one of the main difficulties in the solution of the scattering problem [2,3]. In the literature, an adaptive quadrature method has been proposed for the numerical evaluation of the spectral integral [4,5]. Another method, proposed for the integral evaluation, is based on the Fourier expansion of the reflection coefficient in the kernel [6,7]. This technique allows to compute the homogeneous part of the integral analytically. However, the inhomogeneous part has still to be computed numerically.

In a recent paper, the analytic solution of the spectral integral has been presented, involving the Anger-Weber functions, solutions of the inhomogeneous Bessel differential equation [8]. The solution has been validated by comparison with the adaptive quadrature method and it shows very good behavior especially in the near field where the numerical solution is affected by strong errors. In the present paper, we show further comparisons of the analytic solution with the adaptive numerical method in order to show the advantages in terms of precision and computational speed.

In Section 2, we describe the analytical solution of the RW spectral integral. In Section 3, we present some comparisons between the results obtained with the analytical expression of the RW and those obtained with the quadrature method used up to now in the literature [2]. Finally, in Section 4, the conclusions are drawn.

2. The Generalized Image Principle

The Reflected cylindrical Wave (RW) has been introduced in the scattering by a circular cylinder placed near a plane interface. We consider a reference frame (x,z) and a plane interface, parallel to the z-axis, between a medium 1, with relative permittivity and permeability ε_1 and μ_1, respectively, and a medium 2, with relative permittivity and permeability ε_2 and μ_2, respectively. We normalize the coordinates with respect to the wave number of medium 1: $\xi = k_1 x$ and $\zeta = k_1 z$, see Fig. 1. If we consider a cylindrical wave, of order m propagating in medium 1, the RW can be expressed by the following spectral integral [1]:

$$RW_m(\xi,\zeta) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} R_{E/H}(n_\parallel)\widehat{CW}_m(\xi,n_\parallel)\exp(i\zeta n_\parallel)\mathrm{d}n_\parallel \tag{1}$$

where: $n_\parallel$ is the component parallel to the interface of the wave vector normalized with respect to the wave number of the medium, $R_{E/H}$ is the Fresnel reflection coefficient, for either E or H polarization, respectively, and $\widehat{CW}_m$ is the plane-wave spectrum of a cylindrical function presented in the literature in both the lossless and the lossy case [9,10]. In [8], an analytical evaluation of the integral (1) has been presented. To obtain the solution, it is needed to expand the reflection coefficient $\mathrm{R_{E/H}}$ is series. Unfortunately, the reflection coefficients, in either E or H polarization, cannot be expanded in Laurent series in the whole complex plane because of a polar singularity away from the real axis. On the other hand, we

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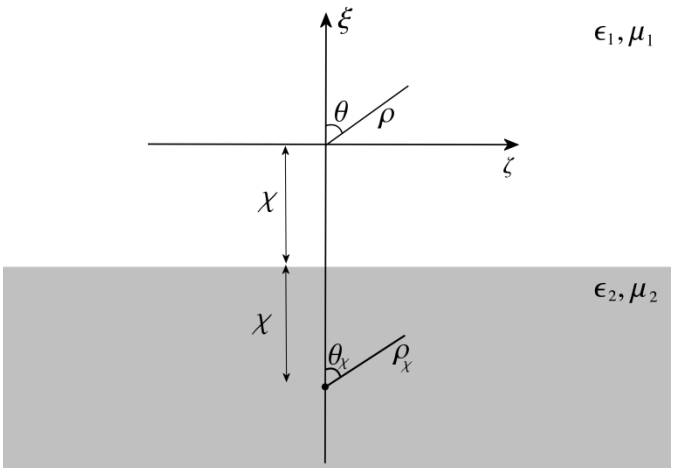

Figure 1: Geometry of the problem.

are interested in the values of the function only on the real axis, where the function does not have singularities. The idea is to perform a conformal mapping of the function to transform the real axis into a circumference away from the singularities. The following conformal mapping has been considered:

$$w = n_\parallel \pm \sqrt{n_\parallel^2 - 1} \qquad \text{and} \qquad n_\parallel = \frac{w^2+1}{2w} \tag{2}$$

This conformal mapping transforms the real axis on the plane of $n_\parallel$, into a semi-circumference of unitary radius in the plane of *w*. It can be seen that the reflection coefficient in the plane of *w* has not singularities in a circle of unitary radius. Therefore, the reflection coefficient can be expanded into Laurent series in the plane of *w*:

$$R_{E/H}(w) = \sum_{k=0}^{\infty} R_k^{E/H} w^k = \sum_{k=0}^{\infty} R_{2k}^{E/H} \exp\left[2ki\arccos(n_\parallel)\right] \tag{3}$$

where $R_k^{E/H}$ are the Laurent coefficients of the Fresnel reflection coefficient:

$$R_k^{E/H} = \frac{1}{2\pi}\int_C \frac{R_{E/H}(w)}{w^{k+1}}\,\mathrm{d}w \tag{4}$$

In the expression (3), we inserted Eq. (2) and we took into account that the Fresnel reflection coefficients, in both polarizations, are even functions, so we considered only the even elements of the series. Making use of the expansion in (3), the integral (1) can be analytically evaluated, obtaining the following expression:

$$RW_m(\rho,\theta) = \sum_{k=0}^{\infty} R_{2k}^{E/H}\left[F_{m-2k}(\rho,\theta) + (-1)^m F_{-m-2k}(\rho,\theta)\right] \tag{5}$$

where ρ and θ are the normalized radius and angle of a polar reference frame on the plane (ξ,ζ), see Fig. 1. The function $F_n(\rho,\theta)$ is defined as follows:

$$F_n(\rho,\theta) = \exp(in\theta)\sum_{k=-\infty}^{+\infty}\left[a_{n,k}(\theta)\mathrm{J}_k(\rho)\right] - iA_n(\rho) \tag{6}$$

where $\mathrm{J}_k(\rho)$ is the Bessel function of the first kind and order k, and $A_n(\rho)$ is the Anger-Weber function of order n [11,12]. Moreover, the coefficients of the series are the following:

$$a_{n,k}(\theta) = \frac{2}{\pi} i^{k-n} \exp\left[i(k+n)\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\mathrm{sinc}\left[(k+n)\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right] \tag{7}$$

The analytical expression (5) is of great importance. In fact, up to now, the only way to evaluate the function (1) was by a quadrature method. Moreover, the expression (6), allows us to physically interpret the analytical solution. In fact, the RW is represented as a superposition of solutions of the homogeneous and inhomogeneous Bessel differential equation. If we consider a cylindrical wave, generated at distance χ above the interface, we can see that the reflected wave is centered at distance χ below the interface. In fact, each elementary wave of the spectrum propagates for a distance χ towards the interface, it is multiplied by the reflection coefficient and it propagates again towards the origin of the reference frame and, after it, continues to propagate in the positive ξ direction. Therefore, we have to consider a shift of -2χ in the argument of the RW, i.e., $\mathrm{RW_m}(\xi-2\chi,\zeta)=\mathrm{RW_m}(\rho_\chi,\theta_\chi)$, see Fig. 1. Therefore, the solution suggests that the interface can be substituted by a cylindrical source placed symmetrically at the origin of the incident cylindrical wave, with respect to the interface, in the same way we do in the presence of an electric charge in front of a Perfectly

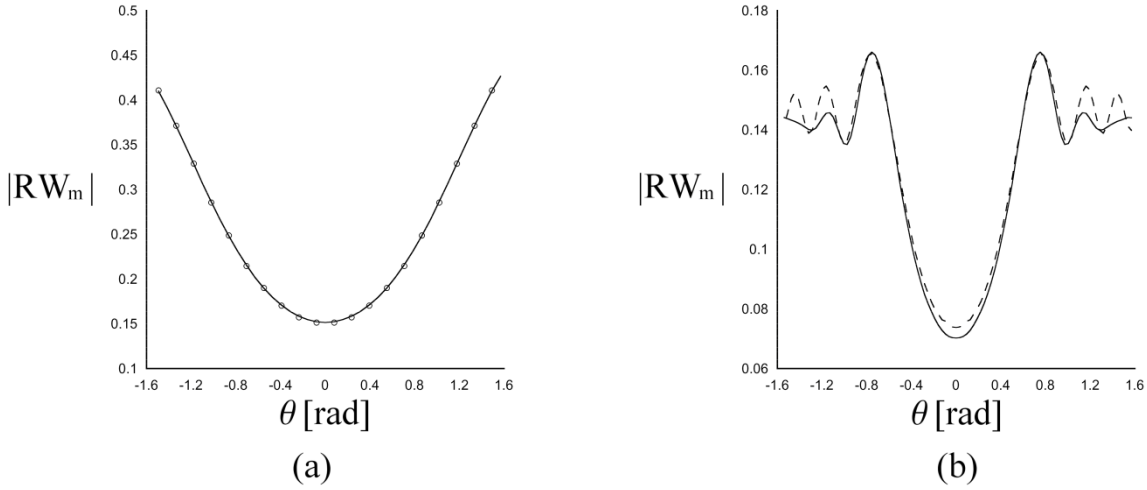

Figure 2: Amplitude of the RW of order *0,* when medium 1 is a vacuum and medium 2 has $\varepsilon_2 = 10$. The RW is computed in $\rho = 5$ (a), and in $\rho = 30$ (b). We applied the analytic formula (5) (solid line) and the quadrature method proposed in the literature [5] (circles (a), dashed line (b)).

Conducting Plane in the well known Image Principle. In this sense, we called this solution a Generalized Image Principle for cylindrical waves. In the following section we will see the advantages of the analytical expression with respect to the quadrature method in terms of numerical precision.

3. Numerical Results

In this section, we present some comparisons between the RW calculated with the analytical formula (5) and with the adaptive quadrature method presented in the literature [5]. In Fig. 2, the amplitude of the RW of order *0* is shown as a function of the angle θ, for a radius $\rho = 5$. We considered the case of an interface between a vacuum and a medium 2 with relative permittivity $\varepsilon_2 = 10$. As it can be seen, the agreement between the analytical formula and the quadrature method is perfect. However, when the radius is far away from the unity, the agreement is not anymore so good. As an example, in Fig. 2b, the same scenario of Fig. 2a is shown, but in the far-field case, i.e., with $\rho = 30$. This time, we see some oscillations of the solution obtained with the quadrature method. Since, up to now, the numerical approach was the only possible for the evaluation of the RW, it is hard to say which is the right behavior between the two curves shown in Fig. 2b. However, the doubts on the correctness of the analytical expression are eliminated when we consider the RW in the near field, i.e., for values of ρ less than unity. In Fig. 3a, the amplitude of the RW is shown in the same scenario of Fig. 2, but for a radius $\rho = 0.6$. Here, we clearly see that the solution obtained with the quadrature method strongly oscillates around the analytical solution. This result means that the numerical errors made with the quadrature method in the near field are not negligible. Moreover, up to now, it was not possible to demonstrate these numerical errors, because it was not possible to evaluate the RW with other techniques. In Fig. 3b, the same amplitude of Fig. 3a is shown, but with a radius $\rho = 0.3$. Here we see that the oscillations increase when the radius decreases. Furthermore, for angles near to $\pm\pi/2$, the oscillations become so large that we were obliged to cancel the extreme values of the function in the graphic because they go out of the scale.

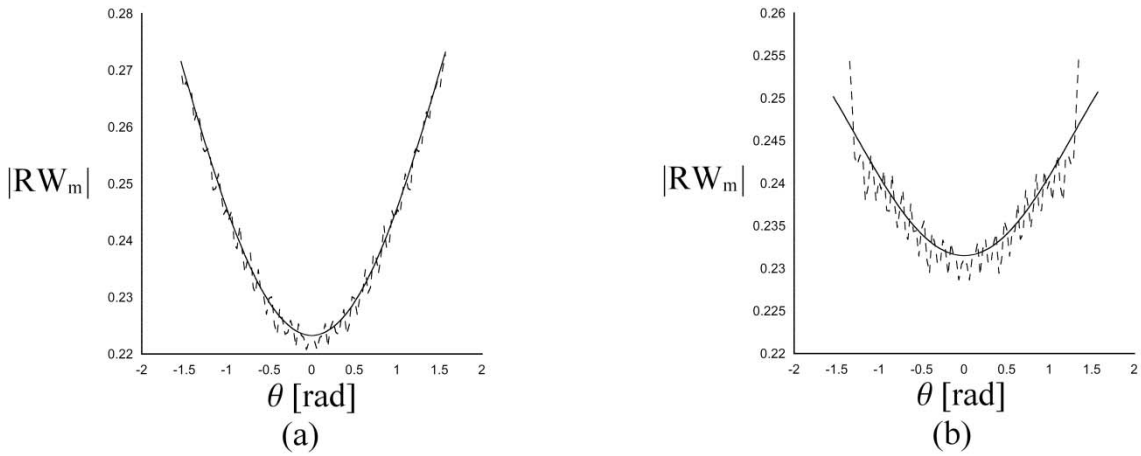

Figure 3: Amplitude of the RW of order *0,* when medium 1 is a vacuum and medium 2 has $\varepsilon_2 = 10$. The RW is computed in $\rho = 0.6$ (a), and in $\rho = 0.3$ (b). We applied the analytic formula (5) (solid line) and the quadrature method proposed in the literature [5] (dashed line).

Finally, we want to make some considerations on the computational speed. In fact, in the scattering problems the RW must be computed many times for different radii and orders. Therefore, the computational speed of the procedure is a crucial parameter in the evaluation of the RW. In all the tests made, for small radii and orders and for large radii and orders, we found that the computational time for the evaluation of the RW is around *0.05* s with the quadrature method and around *0.5* s with the analytical formula. Therefore, the analytical solution is ten times slower than the numerical solution. The reasons of the surprising result are principally two: the first is that the quadrature formula proposed in the literature has been highly optimized. The second reason is that the analytical solution requires the computation of many Bessel functions and Anger-Weber functions. We did not yet optimize the computation of these functions, and we are strongly convinced that with an optimization on the computation of the Bessel and Anger-Weber functions, the analytical solution will reach a computational speed equal to or greater than that of the quadrature method.

4. Conclusions

In this paper, we considered the Generalized Image Principle recently proposed in the literature, to evaluate analytically the reflection of a cylindrical wave by a plane interface between two dielectrics. We briefly described the analytical solution of the spectral integral representing the Reflected cylindrical Wave (RW). We provided some comparisons between the RW evaluated with the analytic formula and with a quadrature method previously presented in the literature. We found a good agreement between the two methods, but we noted that the quadrature method shows some numerical errors for high and low values of the radius. Furthermore, we made some considerations on the computational speed of the two analyzed methods.

We consider that the analytical solution of the RW spectral integral is of great importance from the point of view of both electromagnetic theory and applications. Moreover, the possibility to recognize a physical interpretation of the RW as a wave generated by cylindrical sources below the interface is of great interest.

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