Low-Frequency Multilevel Fast Multipole Algorithm Using an Approximate Diagonalization of the Green's Function

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Abstract

We present an approximate diagonalization of the Green's function to implement a stable multilevel fast multipole algorithm (MLFMA) for low-frequency problems. The diagonalization is based on scaled spherical functions, leading to stable computations of translation operators at all distances and for all frequencies. Similar to the conventional diagonalization, shift operators are expressed in terms of complex exponentials, while radiated and incoming fields are expanded in terms of scaled plane waves. Even though its accuracy is limited, the low-frequency MLFMA developed by using the proposed diagonalization technique provides stable matrix-vector multiplications for arbitrarily low frequencies, while it can easily be implemented via minor modifications on the existing codes.

1. Introduction

Diagonalization of the Green's function is required in the fast multipole method and in its multilevel version, namely, the multilevel fast multipole algorithm (MLFMA) [1]. Unfortunately, the standard diagonalization of the Green's function using plane wave expansions and monopole-to-monopole translation operators suffers from the well-known low-frequency breakdowns [2], which inhibit its application at short distances with respect to wavelength. This is particularly important for low-frequency electrodynamics problems, where objects under investigation are small and their discretizations require much smaller elements with respect to the operating wavelength. Employing a conventional implementation of MLFMA to such a problem leads to many interactions that must be computed directly, leading to quadratic time and memory complexities. In the literature, low-frequency breakdowns are tackled by using alternative diagonalization approaches [3-5] or by simply omitting diagonalizations at subwavelength interactions and computing them via multipole factorizations [6-11]. Unfortunately, most of these approaches need new implementations that must be programmed from scratch, while many of them do not provide the desired efficiency in comparison to plane-wave expansions. Hence, new approaches are still required to solve low-frequency problems both efficiently and accurately.

In this work, we present an approximate diagonalization of the Green's function using scaled spherical functions and plane waves. Even though the diagonalization is approximate and its accuracy is limited, it is valid and stable at arbitrarily short distances. The proposed diagonalization is very similar to the conventional one such that it can be implemented easily via minor modifications on the existing codes of MLFMA. We show that the resulting low-frequency MLFMA implementation provides accurate solutions of low-frequency problems without resorting to fundamental changes in the factorization and diagonalization of the Green's function.

2. An Approximate Diagonalization of the Green's Function

Efficient computations of far-zone interactions in MLFMA require the diagonalization of the Green's function, which can be written in the conventional form as

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{ik}{(4\pi)^2}\int d^2\hat{\boldsymbol{k}}\,\beta(k,\hat{\boldsymbol{k}},\boldsymbol{v})\;\alpha(k,\hat{\boldsymbol{k}},\boldsymbol{w}), \tag{1}$$

where $\boldsymbol{r}-\boldsymbol{r}'=\boldsymbol{w}+\boldsymbol{v}$, $|\boldsymbol{w}|>|\boldsymbol{v}|$, $k=2\pi/\lambda$ is the wavenumber,

$$\beta(k,\hat{\boldsymbol{k}},\boldsymbol{v}) = \exp(ik\hat{\boldsymbol{k}}\cdot\boldsymbol{v}) \tag{2}$$

represents diagonal shift operators, and

$$\alpha(k,\hat{\boldsymbol{k}},\boldsymbol{w}) \approx \sum_{t=0}^{\tau} i^t(2t+1)h_t^{(1)}(kw)P_t(\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{w}}) \tag{3}$$

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represents diagonal (monopole-to-monopole) translation operators [1]. The angular integration in (1) is carried out to account for all plane waves that are used to expand radiated and incoming fields. The summation in (3) is truncated at τ, which depends on the shift and translation distances in terms of wavelength. In an MLFMA implementation, the diagonalization in (1)-(3) is used at different levels involving basis and testing groups of different sizes. In general, aggregation-translation-disaggregation sequences are carried out to perform far-zone interactions on-the-fly, by computing radiated fields of basis groups in terms of plane waves, converting them into incoming fields between far-zone groups, and computing incoming fields at group centers to be received finally by testing functions. For a standard electrodynamics problem involving a large object and its discretization with approximately λ/10 elements, all these operations can be performed in $O(N \log N)$ time while using $O(N \log N)$ memory.

The standard diagonalization in (1)-(3) is prone to stability problems when it is used at short distances, i.e., when $|\boldsymbol{w}|$ and $|\boldsymbol{v}|$ are small with respect to wavelength [2]. Specifically, the spherical Hankel function in (3) becomes very large for small arguments and large orders, leading to a loss of accuracy while they are combined to compute translation operators. In such cases, where the argument is very small, it becomes impossible to obtain a reasonable accuracy, while adding more harmonics leads to a divergence with unbounded translation values. In addition to these problems occurring in the translation stage, it is well known that exponential functions in (2) become incapable of resolving radiation and receiving patterns with a desired level of accuracy. Consequently, the diagonalization in (1)-(3) cannot be used for computing interactions between small groups of basis and testing functions. Unfortunately, low-frequency problems, which involve small objects and details with respect to wavelength, need particularly fine discretizations for accurate modeling of objects, leading to many subwavelength groups. Using a standard diagonalization, such short distance interactions between small groups must be performed directly or by using alternative approaches for the factorization and diagonalization.

The most common approaches for low-frequency problems in the literature involve different diagonalizations of the Green's function, mostly by employing evanescent waves at short distances [3-5]. Alternatively, the diagonalization may be totally avoided [11], while using multipoles for the factorization and using multipole-to-multipole translations to perform far-zone interactions. On the other hand, all these approaches more or less need re-implementations of MLFMA. Even though these techniques are capable of providing highly controllable accuracy, they may be difficult to implement and use, compared to the ordinary implementations of MLFMA. It is therefore desirable to derive new methods for the diagonalization of the Green's function. In this work, we show that an alternative version of the conventional diagonalization, which involves scaled spherical functions that are more stable at low frequencies, is possible. In order to use exponentials (i.e., plane-wave expansions), the new diagonalization requires an approximation that may limit its accuracy. On the other hand, the proposed technique is applicable at arbitrarily low frequencies and it can be implemented via very minor modifications on the existing diagonalization codes.

In general, we consider an alternative diagonalization of the Green's function as

$$g(\boldsymbol{r},\boldsymbol{r}') = \frac{ik}{(4\pi)^2} \int d^2\hat{\boldsymbol{k}} \tilde{\beta}(k,\hat{\boldsymbol{k}},\boldsymbol{v})\ \tilde{\alpha}(k,\hat{\boldsymbol{k}},\boldsymbol{w}), \tag{4}$$

where

$$\tilde{\beta}(k,\hat{\boldsymbol{k}},\boldsymbol{v}) \approx \sum_{t=0}^{\tau} i^t (2t+1) s^{-t} j_t(kv) P_t(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}) \tag{5}$$

and

$$\tilde{\alpha}(k,\hat{\boldsymbol{k}},\boldsymbol{w}) \approx \sum_{t=0}^{\tau} i^t (2t+1) s^t h_t^{(1)}(kw) P_t(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{w}}) \tag{6}$$

are shift and translation operators, respectively. In these operators, spherical Bessel and Hankel functions are scaled with a scaling factor $s < 1$ such that harmonic terms become numerically well balanced for small arguments. We note that (4)-(6) do not involve any approximation, other than the truncations in (5) and (6). In our implementations, where cubic groups with one-box-buffer scheme are used, we select s proportional to the box size with respect to wavelength. In order to facilitate multiple shifts on the radiation and receiving sides, as well as between levels, we further approximate the shift operator as

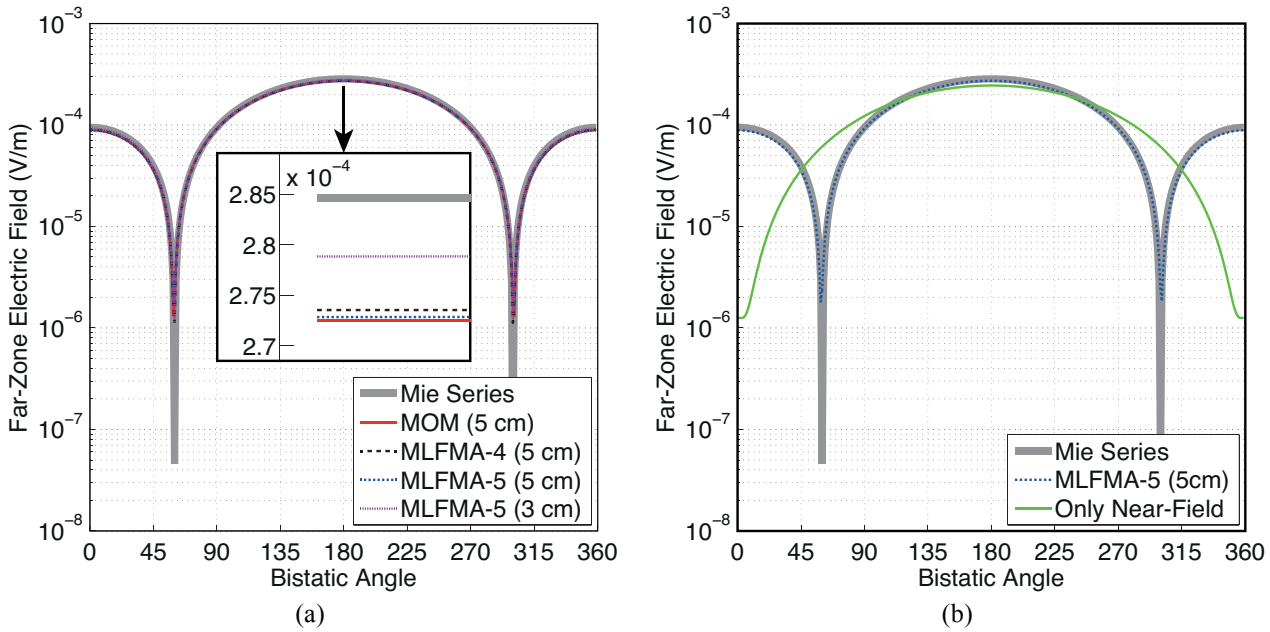
$$\beta(k,\hat{\boldsymbol{k}},\boldsymbol{v}) \approx \sum_{t=0}^{\tau} i^t (2t+1) j_t(kv/s) P_t(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{v}}) = \exp(ik\hat{\boldsymbol{k}} \cdot \boldsymbol{v}/s). \tag{7}$$

The approximation in (7) may deteriorate the controllable accuracy of the new diagonalization. The error involved in this expression can be minimized by selecting $s \gg kv$. Unfortunately, the stabilization of the Hankel function in (6) needs $s \ll kw$, which is contradictory to the criteria for the error minimization in the shift operator. Hence, the scaling factor needs to be carefully selected by considering both stabilization of the spherical functions and the exponential approximation of the shift operator.

3. Numerical Examples

In order to demonstrate the stability and accuracy of the new approximate diagonalization, we consider the solution of a canonical scattering problem involving a sphere of radius 0.3 m at 4 MHz. The radius of the sphere is approximately λ/250, and the problem is discretized with the Rao-Wilton-Glisson functions defined on 5 cm and 3 cm triangles, leading to matrix equations involving 1476 and 4080 unknowns, respectively. The problem is formulated with the magnetic-field integral equation (MFIE) and solved iteratively using MLFMA with four and five levels, corresponding to two and three far-zone levels, respectively. Based on our experiments with the scale factor (not shown here), the truncation number is selected as 9 at all levels. For comparisons, the same problem is also analyzed via Mie-series solutions and with the method of moments (MOM).

Fig. 1(a) depicts the far-zone electric field values on the *z*-*x* plane when the sphere is illuminated by a unit plane wave propagating in the *z* direction with the electric field polarized in the *x* direction. In the plot, 0 and 180 degrees correspond to the forward-scattering and backscattering directions, respectively. It can be observed that computational values obtained with four-level and five-level MLFMA solutions (denoted by MLFMA-4 and MLFMA-5) using 5 cm discretization are in consistent with those obtained with Mie-series and MOM. Backscattering values are further focused in the inset of Fig. 1(a), where approximately 4% error between computational and analytical solutions becomes visible. The major source of this error seems to be the discretization, which can be relatively significant when MFIE is used. In fact, Fig. 1(a) also shows that using a better discretization with 3 cm triangles (and a five-level MLFMA) reduces the error and improves the accuracy of the computational results. As demonstrated in Fig. 1(a), the proposed approximate diagonalization works well, allowing MLFMA solutions of a low-frequency problem that cannot be handled via standard MLFMA implementations.



(a) (b)

Figure 1. Far-zone electric field (V/m) scattered from a perfectly conducting sphere of radius 0.3 m at 4 MHz calculated by using the low-frequency MLFMA, MOM, and Mie-series solutions.

In order to show that accurate computations of far-zone interactions are very critical for accurate scattering computations, even when a low-frequency problem is considered, Fig. 1(b) presents the solution of the same problem using a five-level MLFMA, in comparison to a hypothetical solution that is based on only near-zone interactions. Specifically, MLFMA with the approximate diagonalization is used in the former, whereas only the interactions in the

sparse near-field matrix (based on the five-level tree structure) are considered in the latter. Even though matrix equations derived from MFIE are dominated by the identity term that appears in the near-field matrix, Fig 1(b) clearly shows that far-zone interactions are required to solve the problem with a reasonable accuracy, and these interactions can be performed accurately using the developed diagonalization technique.

4. Conclusion

A novel diagonalization of the Green's function using scaled spherical functions and plane waves is presented for stable solutions of low-frequency problems using MLFMA. The diagonalization is easy to implement on the existing MLFMA solvers, while it provides reasonably accurate solutions at arbitrarily low frequencies.

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