Anomalous Absorption using Hyperbolic Radial Anisotropy

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Abstract

A radially anisotropic sphere with certain ranges of hyperbolic permittivity seems to be lossy even when the losses in the material parameters vanish. This surprising effect is present both in the quasistatic polarizability and in the Mie-scattering solution, and both solutions agree in the long-wavelength limit.

1 Introduction

The static and dynamic response of radially anisotropic (radially uniaxial) spheres, and some generalizations, have been studied in several publications including [1–8]. In this presentation, we continue the work to get a better understanding of the anomalous absorption in spheres with (almost) lossless hyperbolic radial anisotropy. We extend our previously reported quasistatic analysis [7] with some dynamic Mie-scattering results showing that the anomalous absorption is not (only) a static effect. Also [5, 6] briefly mention this absorption, but they mainly concentrate on the enhanced scattering due to plasmonic resonances.

To be more specific, we consider a non-magnetic sphere with radius a and radially anisotropic (RA) permittivity

$$\overline{\overline{\varepsilon}} = \varepsilon_0 \left[\varepsilon_{\text{rad}} \, \mathbf{u}_r \mathbf{u}_r + \varepsilon_{\text{tan}} \left(\mathbf{u}_\theta \mathbf{u}_\theta + \mathbf{u}_\varphi \mathbf{u}_\varphi \right) \right] \tag{1}$$

centered at the origin of the spherical coordinate-system (r, θ, φ). The sphere is illuminated by a time-harmonic ($e^{+j\omega t}$) plane-wave with wave number k. Of special interest is the case when the permittivity components have opposite signs, i.e., when the material is indefinite or hyperbolic.

2 Quasistatic Polarizability

Electrically small particles essentially behave as point-dipole scatterers. In the long-wavelength limit ($ka \to 0$), the magnetic dipole-moment vanishes, since $\mu = \mu_0$ everywhere, and the electric dipole-moment is given by the (quasistatic) polarizability times the incident electric field. If we further normalize the polarizability by ε_0 and the volume of the sphere, we can express the normalized polarizability α of an RA-sphere in terms of an effective relative permittivity ε_{eff} as [7]

$$\alpha = 3 \frac{\varepsilon_{\text{eff}} - 1}{\varepsilon_{\text{eff}} + 2}, \qquad \varepsilon_{\text{eff}} = \frac{\varepsilon_{\text{rad}}}{2} \left(-1 + \sqrt{1 + 8\varepsilon_{\text{tan}}/\varepsilon_{\text{rad}}} \right). \tag{2}$$

Real permittivity components can thus give rise to a complex polarizability. More specifically, the polarizability is complex if $-8 < \varepsilon_{\text{rad}}/\varepsilon_{\text{tan}} < 0$, as illustrated in Figure 1. This result seems totally unphysical at first sight, but a more careful analysis shows that the result is valid in a certain limiting sense for infinitesimally small losses in the material.

The problematic point of the RA-sphere is the origin, where the quasistatic potential can be too singular. To regularize the problem, we can puncture the sphere using a grounded perfectly conducting sphere of radius $b \ll a$. For the punctured RA-sphere, the polarizability can be written in the form (2) with the effective permittivity [7]

$$\varepsilon_{\text{eff}} = \frac{\varepsilon_{\text{rad}}}{2} \left(-1 + \sqrt{1 + 8\varepsilon_{\text{tan}}/\varepsilon_{\text{rad}}} \frac{1 + \left(\frac{b}{a}\right)^{\sqrt{1+8\varepsilon_{\text{tan}}/\varepsilon_{\text{rad}}}}}{1 - \left(\frac{b}{a}\right)^{\sqrt{1+8\varepsilon_{\text{tan}}/\varepsilon_{\text{rad}}}}} \right). \tag{3}$$

The limit $b/a \to 0$ is well defined for all ε_{rad} and ε_{tan}, if we assume that either ε_{rad} or ε_{tan} (or both) contain losses. In the limit, we get $(b/a)^{\sqrt{\cdot}} \to 0$ and the same ε_{eff} as for an intact RA-sphere. Thus, the polarizability (2) is correct also for a punctured RA-sphere in the limit when the puncturing vanishes, if we assume infinitesimally small losses in the material.

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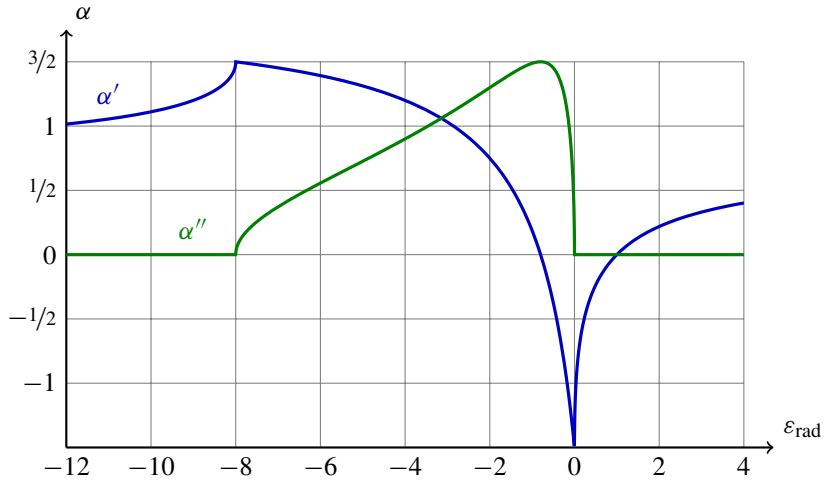

Figure 1: Quasistatic normalized polarizability $\alpha' - \mathrm{j}\alpha''$ for a radially anisotropic sphere with $\varepsilon_{\tan} = 1$. The sphere appears to be lossy when $-8 < \varepsilon_{\mathrm{rad}} < 0$ although the permittivity components are real. Also notice that $|\alpha| = 3/2$ when $-8 < \varepsilon_{\mathrm{rad}} < 0$ in this particular case.

3 Mie Scattering

To solve the scattering of a plane wave from an RA-sphere of arbitrary size, we need the full Mie-series solution. The radial anisotropy can be interpreted as a stretching of the r-coordinate inside the sphere. It turns out that we can use the formulas from [9, Ch. 4], with surprisingly small modifications. The main difference is that the degrees of the spherical Bessel functions inside the sphere are

$$\nu = \nu(n) = -\frac{1}{2} + \sqrt{n(n+1)\frac{\varepsilon_{\tan}}{\varepsilon_{\mathrm{rad}}} + \frac{1}{4}}, \qquad n = 1, 2, \ldots \tag{4}$$

In particular, and in agreement with [5, 6], we get the scattering coefficients

$$a_n = \frac{m\psi_\nu(mx)\psi_n'(x) - \psi_n(x)\psi_\nu'(mx)}{m\psi_\nu(mx)\xi_n'(x) - \xi_n(x)\psi_\nu'(mx)}, \tag{5}$$

$$b_n = \frac{\psi_n(mx)\psi_n'(x) - m\psi_n(x)\psi_n'(mx)}{\psi_n(mx)\xi_n'(x) - m\xi_n(x)\psi_n'(mx)}, \tag{6}$$

where $x = ka$ is the size parameter, $m = \sqrt{\varepsilon_{\tan}}$ is the (tangential) refractive index, and ψ and ξ are the Riccati–Bessel functions $\psi_\nu(x) = x j_\nu(x)$ and $\xi_\nu(x) = x h_\nu^{(2)}(x)$. (Notice the trivial but important difference in time-convention.) To study the overall response of the sphere, it is convenient to look at the scattering, extinction and absorption efficiencies

$$Q_{\mathrm{sca}} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\left(|a_n|^2 + |b_n|^2\right), \tag{7}$$

$$Q_{\mathrm{ext}} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\,\mathrm{Re}\{a_n + b_n\}, \tag{8}$$

$$Q_{\mathrm{abs}} = Q_{\mathrm{ext}} - Q_{\mathrm{sca}}. \tag{9}$$

The absorption efficiency should be zero for lossless (real) material parameters, and this is also the case for RA-spheres with $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} > 0$. However, we get anomalous absorption for hyperbolic radial anisotropy as shown in Figure 2. The absorption is related to the electrical multipole coefficients a_n. The degree (4) is complex when $-4n(n+1) < \varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} < 0$, and this gives rise to effective losses in the corresponding electric multipole of degree n. The most significant absorption is given by the electrical dipole term a_1, exactly as predicted by the quasistatic approximation, but also the higher order multipoles contribute to the losses. Notice how the limits $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} = -8$ ($n = 1$) and $\varepsilon_{\mathrm{rad}}/\varepsilon_{\tan} = -24$ ($n = 2$) stand out in Figure 2.

The scattering efficiency of a small hyperbolic RA-sphere is typically small compared with the absorption. For instance, for $x \leq 1/10$ in Figure 2, $\max(Q_{\mathrm{sca}}) \ll \max(Q_{\mathrm{abs}})$. Small RA-spheres can also exhibit strong scattering, in a similar way as small homogeneous spheres with negative permittivity. As shown in [5, 6], these resonances can

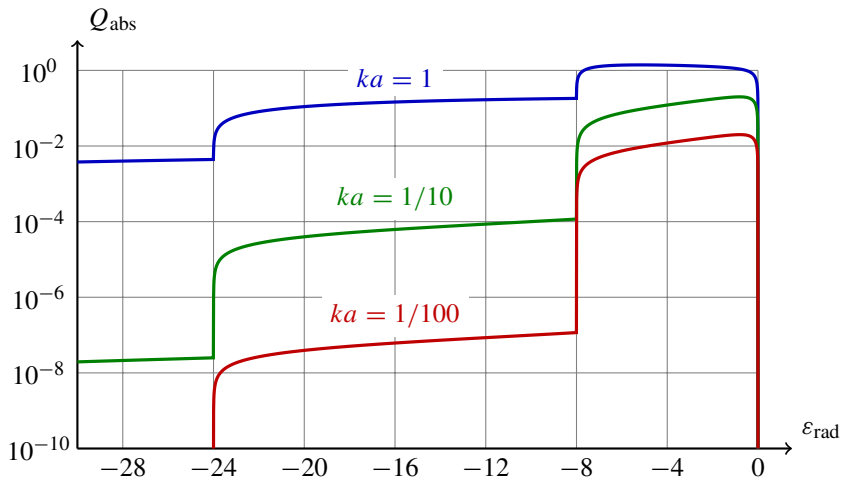

Figure 2: Absoption efficiency for a non-magnetic radially anisotropic sphere with $\varepsilon_{\text{tan}} = 1$ for different size parameters $x = ka$. The absorption is exactly zero when $\varepsilon_{\text{rad}} > 0$, and the absorption also vanishes when $\varepsilon_{\text{rad}} \to -\infty$. The absorption is dominated by the dipole term a_1 when $-8 < \varepsilon_{\text{rad}} < 0$, while the higher order terms $a_2, a_3, \ldots$ are smaller but contributes over larger ranges of ε_{rad}.

be shifted by adjusting ε_{rad} and ε_{tan}. Strong scattering due to plasmonic resonances is, however, mainly found when both ε_{rad} and ε_{tan} are negative, while the anomalous absorption needs ε_{rad} and ε_{tan} of different signs.

Using the Taylor-series expansion of the Riccati–Bessel functions at $x = 0$, we get

$$a_1 = \mathrm{j}\frac{2}{9}\alpha x^3 + \mathrm{O}(x^5), \tag{10}$$

where α is the quasistatic polarizability (2), and all the other scattering coefficients are of the order x^5 or smaller. Thus, the quasistatic approximation and the Mie-series solution agree in the long-wavelength limit.

4 Conclusions

The idea of getting losses from lossless material does not make sense as such, but both the dynamic and quasistatic response of a hyperbolic RA-sphere can give effective absorption in the limit when the losses in the material parameters vanish. Instead of dismissing the result as a mathematical anomaly, we carefully considered the quasistatic case as the limit of a punctured RA-sphere with infinitesimally small losses. Moreover, we showed that the Mie-scattering solution agrees very well with the quasistatic prediction. In the long-wavelength limit, the results agree exactly. It is also interesting to notice that the range $-8 < \varepsilon_{\text{rad}}/\varepsilon_{\text{tan}} < 0$, where the electric dipole term dominates the absorption, is the same as in the quasistatic solution, regardless of the size of the sphere.

Acknowledgments

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