Equivocation Performance of Linear Wireless Physical Layer Network Coding

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Abstract—We consider linear mapping functions for wireless physical layer network coding (WPNC), using a generalised description of linear mapping, and considering alternative algebraic structures within which a linear mapping can be defined. We discuss the requirements on a linear mapping for WPNC, particularly in terms of ensuring unambiguous decoding, and show that this requires that the coefficients of the linear mapping have an inverse within the coefficient set. We then evaluate the information lost on the channel in terms of the equivocation for arbitrary relative fade states, as well as singular fade states. This is then used to determine outage probability and outage capacity for LPNC systems.

I. INTRODUCTION

In Wireless Physical-layer Network Coding (WPNC) a node in a network which receives two or more incoming data streams forwards a function of both incoming streams. The relaying node attempts to decode the function directly from the received signal, rather than decoding each stream separately and then applying the network code function which we refer to as a hierarchical network code (HNC) map [1]. In [2] we discussed linear functions which may be used for the HNC map, introducing functions based on algebraic rings, fields, and on binary matrices. This leads to Linear Physical-layer Network Coding (LPNC). In this paper we restrict our attention to functions based on integer rings. In WPNC the effect of the wireless channel is fundamental: we have to take account of the fading coefficients of the channels between nodes. In particular we find that some fade coefficients of incoming channels at a node will cause certain mapping functions to fail (in a sense we will explore below), even if none of the channels is subject to severe amplitude fading. We note however that it is the relative rather than the absolute values of the coefficients that give rise to this effect, and we refer to these relative fading coefficient values as singular fade states. However it is clear that there are other fade states of the channel which will give rise to information loss, especially in the presence of noise. In this paper we consider the full range of possible relative fade states, and evaluate the information lost on the channel due to this combination of fading and noise. As in [2] we evaluate performance in terms of the equivocation between the received signal at the relay and the network-coded data. This leads to an estimate of the information lost in reaching the relay, which in turn gives the maximum rate which can be transmitted via the relay. Here we calculate the equivocation for arbitrary fade states, rather than just singular fade states. This then allows us to determine outage probability for given code rate.

The outline of the remainder of the paper is as follows. In the next section we give the system model, defining LPNC based on ring algebra, and also singular fading. We then discuss the use of equivocation to evaluate LPNC performance, and especially to calculate outage capacities. Finally we give numerical results for the equivocation against relative fade coefficient for various ring-based LPNC schemes, and for the outage against SNR, and conclude the paper.

II. SYSTEM MODEL

Our focus in this paper is on the information lost in the source-relay links of a system involving LPNC, rather than the end-to-end performance of a given network. However for simplicity we use the two-way relay channel (TWRC) as an illustration: see Figure 1.

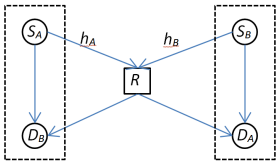

Figure 1 Two-way relay channel

Here two terminals exchange information via a relay: sources S_A and S_B transmit simultaneously to the relay in the first (multiple access, or MAC) phase, then the relay broadcasts a network coded combination of the two source symbols to the destinations in the second (broadcast, or BC) phase. The destination for S_A information, D_A, is collocated with S_B, and has access to S_B which can be used to help decode S_A from the network coded data from the relay. We refer to the former as *hierarchical side information* (HSI) which is combined with the *hierarchical information* (HI) from the relay to obtain the required data. The same of course applies at D_B. The relay attempts to decode a linear function of the source symbols s_A, s_B:

$$f(s_A, s_B) = \alpha_A \odot s_A \oplus \alpha_B \odot s_B \tag{1}$$

where α_A, α_B are coefficients and the operators $\oplus$ and $\odot$ denote operations we will refer to as 'addition' and 'multiplication' respectively. In this paper we assume that all symbols and coefficients are members of algebraic rings. We will restrict our attention

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to QPSK, and hence the symbols s_A, s_B are from the ring $\mathcal{R}(4) = \{0\ldots3\}$. However the coefficients and the output of the function may in general belong to a larger ring $\mathcal{R}(q) = \{0\ldots q\text{-}1\}$, $q \geq 4$. Addition and multiplication are then defined as addition and multiplication modulo q. In [2] we show that provided the coefficients have inverses in the ring, then unambiguous decoding is possible, provided appropriate HSI is available. This in turn requires that the coefficients are relatively prime to q. We assume that the relay can choose the network coding function according to the relative fade coefficient, and in this paper we assume that the coefficients are chosen to allow unambiguous decoding, as well as to optimise performance for the given fade state. The elements in the rings of sizes 4 to 8 to which this applies are: 1) $\mathcal{R}(4)=\{1,3\}$; 2) $\mathcal{R}(5)=\{1,2,3,4\}$; 3) $\mathcal{R}(6)=\{1,5\}$; 4) $\mathcal{R}(7)=\{1,2,3,4,5,6\}$; 5) $\mathcal{R}(8)=\{1,2,3,5,7\}$. Here, for simplification, we will assume that the BC phase transfers the HI perfectly from the relay to both destinations, and we concentrate on the MAC phase. We assume that the channels in the MAC phase from sources A and B to the relay are subject to frequency-flat and time invariant fading, with fade coefficients h_A and h_B respectively. Thus the signal r at the relay in the MAC phase is given by:

$$r = h_A x_A + h_B x_B + n = r_0 + n \tag{2}$$

where n is the noise at the relay, and $r_0 = r_0(s_A,s_B)$ is the noiseless *superimposed signal* at the relay. $x_A = \mathcal{M}(s_A)$ and $x_B = \mathcal{M}(s_B)$ are the transmitted signals from the two sources, where $\mathcal{M}$ denotes a modulation mapping function and $s_A, s_B \in \mathcal{S}$ denote the source symbols. $|\mathcal{S}| = M$ then denotes the modulation order. As mentioned above, here $M = 4$.

The task of the relay is to decode from r, not s_A and s_B, but $f(s_A,s_B)$, as given in (1), We assume that $f(s_A,s_B) \in \mathcal{S}_f, |S_f| = q$. In addition to the requirement for unambiguous recoding, the coefficients in (1) should be chosen to minimise the uncertainty of $f(s_A,s_B)$ given r, that is, the equivocation at the relay. In general of course each pair of symbols (s_A,s_B) results in a different value of r_0, so that the cardinality of the received constellation at the relay is M^2. However there are certain values of h_A and h_B for which some of these constellation points are coincident. We refer to these sets of values as *singular fade states*.

A singular fade state occurs if and only if:

$$h_A x_A + h_B x_B = h_A x'_A + h_B x_B \Rightarrow x_A + \frac{h_B}{h_A} x_B = x'_A + \frac{h_B}{h_A} x'_B \tag{3}$$

Hence it is the *relative fading coefficient* $h_B/h_A = h_{re}$ that determines singular fading, and we note also that the scenario is symmetrical in that the role of h_A and h_B can be reversed, so that we can assume without loss of generality that $|h_B| \leq |h_A|$, i.e. that $|h_{re}| \leq 1$. Then (3) can be rewritten as:

$$h_{re} = \frac{x'_A - x_A}{x_B - x'_B} \tag{4}$$

In this paper we will confine our detailed consideration to QPSK modulation, whose constellation is defined by:

$$x = \mathcal{M}(s \in \mathcal{S}) \in \frac{a_0 + a_1 j}{\sqrt{2}},\ a_k = 2i_k - 1, i_k \in 0,1,\ k = 0,1 \tag{5}$$

Throughout the paper we assume a Gray code labelling, i.e. that i_0 and i_1 form the bits of a 2 bit constellation label, which can equivalently be represented as an integer $i = i_0 + 2i_1 \in \{0,1,2,3\}$. Hence for QPSK singular fade states occur for:

$$h_{re} = \left\{\pm 1, \pm j, \frac{\pm 1 \pm j}{2}, 0\right\} \tag{6}$$

Of these $h_{re} = 0$ is in some ways a special case, since it implies that one of the channels is completely faded, and hence the two-way relay system cannot operate at all.

III. Performance

Information is of course inevitably lost when one channel or both are severely faded. It may also be lost in other singular fade states. The effect of singular fading is that two or more pairs of source symbols result in the same received signal at the relay. We may describe this as a "clash" of symbol pairs. However if these symbol pairs correspond to the same network code symbol, and the network code mapping is unambiguously decodable, then the source data is recoverable without loss. We say that the corresponding singular fade state has been resolved for the set of symbol pairs in question. The ideal case is that for each singular fade state a network code mapping can be found which resolves it for all sets of symbol pairs which would otherwise result in the

same received signal. The relay can then adapt its mapping to the channel state. We may note, as in [2], however, that all is not lost if a particular fade state cannot be fully resolved, for all symbol pairs, in that information may still be conveyed in a singular fade state, because not all symbol pairs will clash. We also considered the case in which those that do clash still give some information about the network coded symbol. However, because to make use of that information would require detailed information about the clash table at the relay to be passed on to the destination, here we do not take this into account.

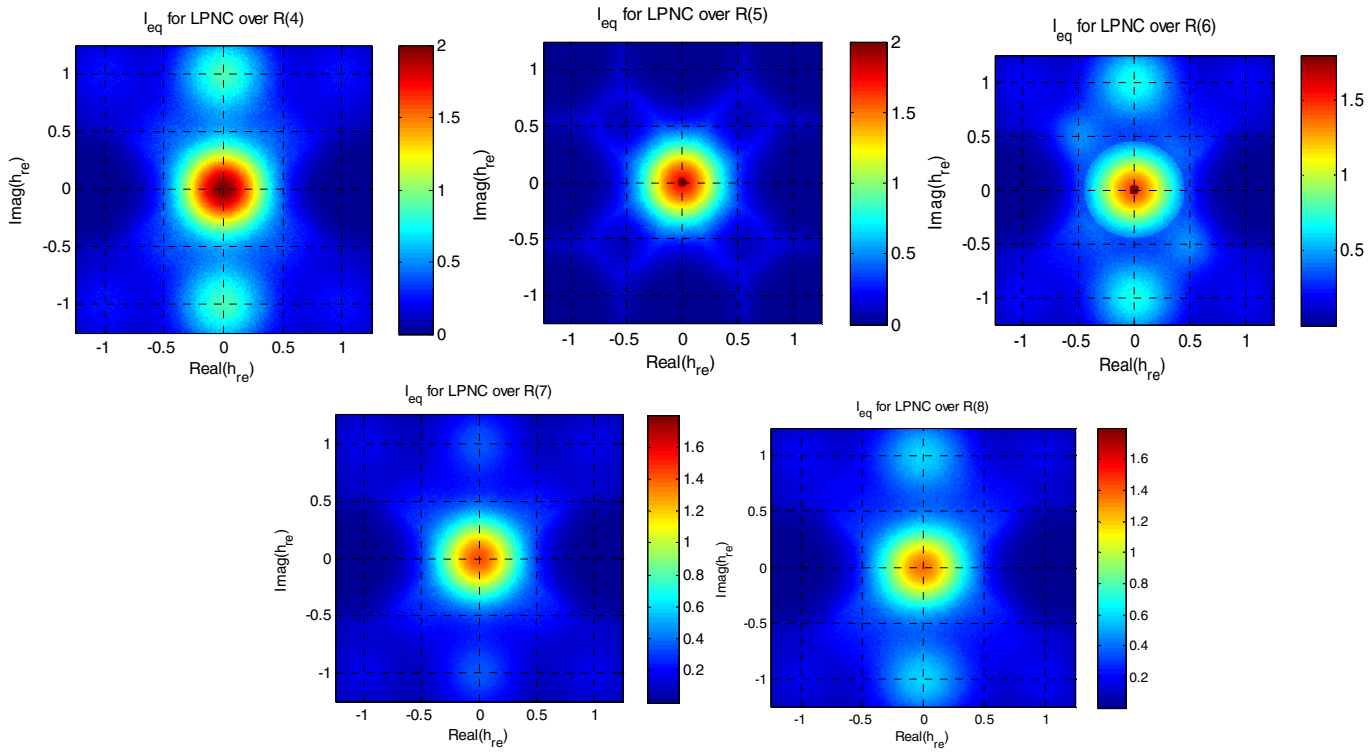

Figure 2 Equivocation against relative fade coefficient h_{re} for rings from $\mathcal{R}(4)$ to $\mathcal{R}(8)$, SNR = 10 dB

In the presence of noise information is also lost when received signals are close together, i.e. when the fade state is close to singular. We measure the information lost by means of the equivocation between the received signal at the relay and the network coded function:

$$I_{eq} = H(F|R) = \sum_{f \in S_f} P(f) \int_{r \in \mathbb{C}} p(r|f) \log_2 \left(\frac{1}{P(f|r)} \right) dr = \sum_{f \in S_f} P(f) \int_{r \in \mathbb{C}} p(r|f) \log_2 \left(\frac{p(r)}{p(r|f) P(f)} \right) dr \tag{7}$$

where the distribution of f:

$$P(f) = \sum_{s_A, s_B : f = f(s_A, s_B)} P(s_A) P(s_B) = \frac{\text{num}\left(f = f(s_A, s_B)\right)}{M^2} \tag{8}$$

in which num($f = f(s_A, s_B)$) denotes the number of pairs (s_A, s_B) for which $f = f(s_A, s_B)$, for given f. Note that for $q > M$ this distribution is in general not uniform. The distribution of r is:

$$p(r) = \sum_{s_A, s_B} p_{\mathcal{N}} \left(\frac{r - r_0(s_A, s_B)}{\sigma} \right) P(s_A) P(s_B) = \frac{1}{M^2} \sum_{s_A, s_B} p_{\mathcal{N}} \left(\frac{r - r_0(s_A, s_B)}{\sigma} \right) \tag{9}$$

where the complex normal distribution:

$$p_{\mathcal{N}}(z) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{|z|^2}{2} \right) \quad \text{and} \quad p(r|f) = \sum_{s_A, s_B : f = f(s_A, s_B)} p_{\mathcal{N}} \left(\frac{r - r_0(s_A, s_B)}{\sigma} \right) \tag{10}$$

Figure 2 shows the resulting equivocation as a function of the relative fade coefficient h_{re} for the rings $\mathcal{R}(4)$ to $\mathcal{R}(8)$. We note, of course, that there is a maximum equivocation where $|h_{re}| = 0$, since one of the channels fades completely. We note also that there are maxima of equivocation around the singular fade values, where this singular fading cannot be resolved by the network code. Comparing the different rings, we note that: 1) the larger rings have lower equivocation; 2) prime number rings (i.e. $\mathcal{R}(5)$ and $\mathcal{R}(7)$,

which are also fields) also have lower equivocation than non-prime rings of similar size; and 3) only $\mathcal{R}(5)$ can resolve all singular fade states, except $|h_{re}| = 0$. This compares with the mapping in [3], except that a non-linear mapping is used there.

The mutual information is then the source entropy, i.e. $\log_2(M)$, minus the equivocation. Hence we can find the outage capacity for given transmission rate R_{th} by finding the probability over the distribution of h_{re} that the mutual information is less that R_{th}

$$P_{out} = \int_{h_{re} \in \mathbb{C}} p(h_{re}) \mathcal{U}\left(\log_2(M) - I_{eq} < R_{th}\right) dh_{re} \tag{11}$$

where $\mathcal{U}(z > z_t)$ takes the value 1 for $z > z_t$, 0 otherwise.

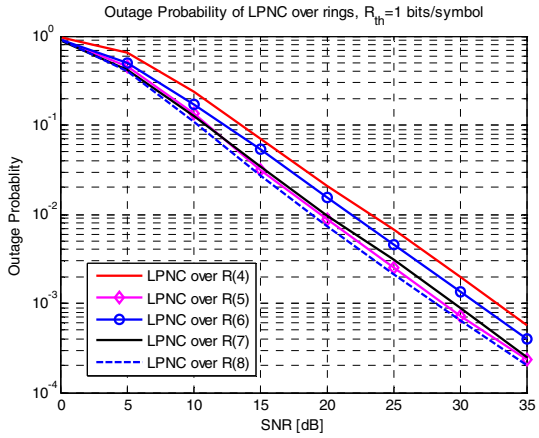

Figure 3 Outage probability against SNR for different rings, transmission rate 1 bit/symbol

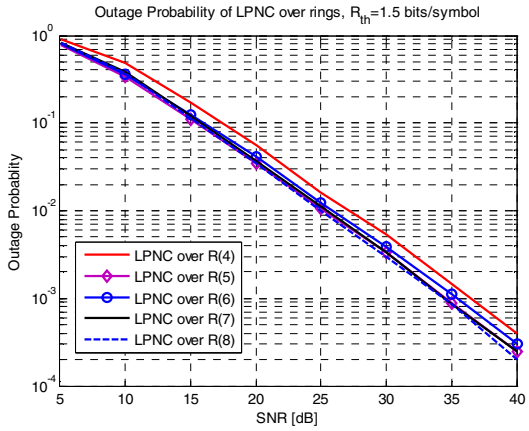

Figure 4 Outage probability against SNR for different rings, transmission rate 1.5 bits/symbol

Figure 3 and Figure 4 show the resulting outage probability results, for rates of 1 and 1.5 bits/symbol respectively, corresponding to code rates at the source of ½ and ¾ respectively. Comparing the different rings, we note the same comparison as above. The lowest outage is given by $\mathcal{R}(8)$ – but note that we have assumed an ideal lossless BC phase, while in practice $\mathcal{R}(8)$ will double the cardinality transmitted on the BC, which is likely to degrade its performance. The performance of $\mathcal{R}(5)$, which as we have seen can resolve all non-trivial singular fade states, is only fractionally poorer. For rate ½ codes the difference between these rings and the minimal mapping, $\mathcal{R}(4)$, is between 4 and 5 dB; for rate ¾ the difference is less. The implementation of these schemes however relies on error correction coding, which is not straightforward for non-minimal mappings, especially where q is not a power of 2. Further work is required on this.

IV. Conclusions

We have examined the performance of linear physical layer network coding, using linear mapping functions based on rings over integers. We have determined the equivocation as a function of the relative fade coefficient, and hence have determined the outage probability for the scheme. We show a significant improvement in outage performance, of 4-5 dB, through the use of non-minimal mappings. However these have the disadvantage that they increase the cardinality required in the broadcast phase of the relay operation, potentially degrading the outage capacity at that point, and also that FEC coding is not easily implemented.

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