Robust Minimum Variance Beamforming

***Qiubin Gao*[1*], *Wei Shi*[2], *Shaohui Sun*[1]**

[1]State Key Laboratory of Wireless Mobile Communication, China Academy of Telecommunication Technology
Beijing 100085, China
[2]Department of Automation, Tsinghua University

Email: gaoqiubin@catt.cn

Abstract

Robust adaptive beamforming is a key issue in array applications where uncertainties about the steering vector of interest exist. This paper considers the worst-case optimization-based robust beamformer problem. The closed-form optimal solution is given for any convex uncertainty set. Numerical examples of robust beamforming against signal look direction mismatches are presented to demonstrate the effectiveness of the robust beamformer.

1. Introduction

Adaptive beamforming plays an important role in a number of applications such as wireless communication, microphone array speech processing, radar, sonar systems and so on. The Capon beamformer selects adaptively the weight vector to minimize the array output power while keeping the desired signal from distortion [1]. However, signal look direction mismatches and array manifold mismodellings often exist in practical systems, and the performance of the conventional beamforming may substantially degrade [2]. Many approaches have been proposed to increase the robustness against array steering vector errors during the past decades. For example, linearly constrained minimum variance beamforming (LCMV) [3], Bayesian approach [4], and recently developed convex optimization using second-order cone programming (SOCP) [5]–[7].

Worst-case robust beamforming is recently proposed and regarded as a promising approach. It is designed to optimize the worst-case performance among all possible steering vectors in the uncertainty set. In previous works, the uncertainty set is described as a convex region and the optimization problem is cast as a convex optimization problem. The uncertainty set is modelled as sphere [5], an ellipsoid [7] or a polyhedral cone [8]. These modelling methods can not fully exploit the structure of the uncertainty set and it may sometimes lead to an infeasible design.
In this paper, we investigate the worst-case performance optimization problem, and give the optimal weight vector in closed form, whenever the uncertainty region is convex. Interestingly, this solution turns out to be very similar to that of Capon's method.

2. Background and formulations

Consider an array consisting of M sensors. A far-field narrowband source $s(t)$ impinges on the array from angle θ. Denote the steering vector as $\boldsymbol{a}(\theta) \in \mathbb{C}^M$. The sampled array observation vector $\boldsymbol{x}(k) \in \mathbb{C}^M$ is then described as

$$\boldsymbol{x}(k) = s(k)\boldsymbol{a}(\theta) + \boldsymbol{i}(k) + \boldsymbol{n}(k) \tag{1}$$

where $s(k)$, $\boldsymbol{i}(k)$, and $\boldsymbol{n}(k)$ are the desired signal, interference, and noise, respectively; and k is the time index. The desired signal and interferers are assumed to be uncorrelated and stationary. The output of a narrowband beamformer is the given by

$$\boldsymbol{y}(k) = \boldsymbol{w}^H \boldsymbol{x}(k) \tag{2}$$

where $\boldsymbol{w} \in \mathbb{C}^M$ is the beamformer weights, and $(\cdot)^H$ represents Hermitian transpose.
The optimal weight vector that maximizes SINR is given by

$$\boldsymbol{w}_{\text{opt}} = \alpha \boldsymbol{R}_{i+n}^{-1} \boldsymbol{a}(\theta) \tag{3}$$

where $\alpha \neq 0$ is an arbitrary number that does not affect SINR, σ_s^2 is power of the desired signal, and $\boldsymbol{R}_{i+n}$is the $M \times M$ interference-plus-noise covariance matrix.

The minimum variance distortionless response (MVDR) beamformer is designed by

$$\begin{aligned} &\min_{\boldsymbol{w}} \ \boldsymbol{w}^H \boldsymbol{R}_x \boldsymbol{w} \\ &\text{s.t. } \boldsymbol{w}^H \boldsymbol{a}(\theta) = 1 \end{aligned} \tag{4}$$

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where

$$\boldsymbol{R}_x = \mathrm{E}\left\{\boldsymbol{x}(k)\boldsymbol{x}(k)^H\right\} = \sigma_s^2\boldsymbol{a}(\theta)\boldsymbol{a}^H(\theta) + \boldsymbol{R}_{i+n} \tag{5}$$

It is also called Capon's beamforming. The solution is given by

$$\boldsymbol{w} = \frac{\boldsymbol{R}_x^{-1}\boldsymbol{a}(\theta)}{\boldsymbol{a}^H(\theta)\boldsymbol{R}_x^{-1}\boldsymbol{a}(\theta)} \tag{6}$$

It is well known that if $\boldsymbol{R}_x$ and $\boldsymbol{a}(\theta)$ are known accurately, (6) is identical to (3). In practice, $\boldsymbol{R}_x$ is unavailable and replaced by its estimate

$$\hat{\boldsymbol{R}}_x = \frac{1}{N}\sum_{n=1}^{N}\boldsymbol{x}(n)\boldsymbol{x}^H(n), \tag{7}$$

where N is the number of snapshots. Therefore, the exact values of $\boldsymbol{a}(\theta)$ and $\boldsymbol{R}_x$ are not available. Instead, they are known to belong to a set which is called uncertainty set.

In order to exploit the structure knowledge of the uncertainty set, robust minimum variance beamforming (RMVB) problem has been proposed in recent years. As a variation of MVDR, it minimizes the output variance while maintaining great-than-unity gain for all steering vectors in the uncertainty set, that is,

$$\begin{aligned} &\min_{\boldsymbol{w}}\ \boldsymbol{w}^H\boldsymbol{R}_x\boldsymbol{w} \\ &\text{s.t. } |\boldsymbol{w}^H\boldsymbol{a}| \geq 1, \forall\, \boldsymbol{a} \in \Delta_{\boldsymbol{a}} \end{aligned} \tag{8}$$

where $\Delta_{\boldsymbol{a}}$ denotes the uncertainty set of steering vector. Vorobyov *et al.* [5] considers arbitrary unknown but norm-bounded mismatches, i.e., spherical uncertainty set $\Delta_{\boldsymbol{a}} = \{\boldsymbol{a} \,|\, \|\boldsymbol{a} - \boldsymbol{a}_{\text{nom}}\| \leq r\}$ where $\boldsymbol{a}_{\text{nom}}$ is the nominal steering vector, and r is the radius of the sphere. It is shown that (8) with the spherical uncertainty set has a solution expressed in the form of diagonal loading, that is,

$$\boldsymbol{w} = \alpha\left(\hat{\boldsymbol{R}}_x + \lambda r^2\boldsymbol{I}\right)^{-1}\boldsymbol{a}_{\text{nom}}$$

where α is a scaling factor and λ is the Lagrange multiplier.
The uncertainty set $\Delta_{\boldsymbol{a}}$ in [8] is covered by a polyhedral cone, and in [6], [7] it is described as an ellipsoid $\Delta_{\boldsymbol{a}} = \left\{\boldsymbol{a} \,|\, (\boldsymbol{a} - \boldsymbol{a}_{\text{nom}})^H\boldsymbol{P}^{-1}(\boldsymbol{a} - \boldsymbol{a}_{\text{nom}}) \leq 1\right\}$, where the vector $\boldsymbol{a}_{\text{nom}}$ is the center of the ellipsoid and Hermitian matrix $\boldsymbol{P}$ is the configuration matrix.

It is shown in [5]–[7] that when $\Delta_{\boldsymbol{a}}$ is a sphere or ellipsoid, problem (8) can be cast as SOCP. However, for other shapes of uncertainty set, SOCP is not applicable. It can be shown that the RMVB problem (8) is equivalent to

$$\max_{\boldsymbol{w}}\ \min_{\boldsymbol{a}\in\boldsymbol{\Delta}_{\boldsymbol{a}}}\quad \frac{|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}} \tag{9}$$

Considering uncertainties both in the steering vector and noise covariance matrix, the optimization (9) could be generalized to

$$\max_{\boldsymbol{w}}\ \min_{(\boldsymbol{a},\boldsymbol{R})\in\Delta}\quad \mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R}) \tag{10}$$

where $\mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R})$ is defined as

$$\mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R}) = \frac{|\boldsymbol{w}^H\boldsymbol{a}|^2}{\boldsymbol{w}^H\boldsymbol{R}\boldsymbol{w}}, \boldsymbol{w} \in \mathbb{C}^M, \boldsymbol{a} \in \mathbb{C}^M, \boldsymbol{R} \in \mathbb{H}_{++}^M \tag{11}$$

The problem maximizes the worst-case SINR. It seeks a filter $\boldsymbol{w}$ that maximizes the minimum SINR over the uncertainty set of signal $\boldsymbol{a}$ and noise covariance matrix $\boldsymbol{R}$.

3. Solution for Convex Uncertainty

A saddle point $(\boldsymbol{w}_*, \boldsymbol{a}_*, \boldsymbol{R}_*)$ exists for $\mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R})$ is to say

$$\begin{aligned} &\mathrm{S}(\boldsymbol{w};\boldsymbol{a}_*,\boldsymbol{R}_*) \leq \mathrm{S}(\boldsymbol{w}_*,\boldsymbol{a}_*,\boldsymbol{R}_*) \leq \mathrm{S}(\boldsymbol{w}_*;\boldsymbol{a},\boldsymbol{R}) \\ &\forall\, \boldsymbol{w} \in \mathbb{C}^M \setminus \{\boldsymbol{0}\}, \quad \forall\, (\boldsymbol{a},\boldsymbol{R}) \in \Delta \end{aligned} \tag{12}$$

A direct consequence of the existence of saddle point is

$$\max_{\boldsymbol{w}}\ \min_{(\boldsymbol{a},\boldsymbol{R})\in\Delta} \mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R}) = \min_{(\boldsymbol{a},\boldsymbol{R})\in\Delta}\ \max_{\boldsymbol{w}} \mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R}) \tag{13}$$

Consequently, the max-min problem can be solved by solving the min-max problem. The inner maximization of the min-max problem is easily solved. For given $\boldsymbol{a}$ and $\boldsymbol{R}$, $\max_{\boldsymbol{w}} \mathrm{S}(\boldsymbol{w};\boldsymbol{a},\boldsymbol{R}) = \boldsymbol{a}^H\boldsymbol{R}^{-1}\boldsymbol{a}$ with $\boldsymbol{w}_{\text{opt}} = \alpha\boldsymbol{R}^{-1}\boldsymbol{a}$ where

$\alpha \neq 0$ is an arbitrary complex number. Therefore, if the existence of the saddle point is guaranteed, the solution to problem (10) is given by the following minimization problem

$$\begin{cases} (\boldsymbol{a}_{\mathrm{wc}}, \boldsymbol{R}_{\mathrm{wc}}) = \arg \min_{(\boldsymbol{a},\boldsymbol{R})\in\Delta} \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \boldsymbol{w}_* = \alpha \boldsymbol{R}_{\mathrm{wc}}^{-1} \boldsymbol{a}_{\mathrm{wc}}, \quad \alpha \in \mathbb{C} \setminus \{0\} \end{cases} \tag{14}$$

in which $\boldsymbol{a}_{\mathrm{wc}}$ and $\boldsymbol{R}_{\mathrm{wc}}$ denote the worst-case steering vector and covariance matrix.

It is easily verified that the SINR functional $\mathrm{S}(\boldsymbol{w}; \boldsymbol{a}, \boldsymbol{R})$ is convex over $(\boldsymbol{a}, \boldsymbol{R})$. According to results in [9], if the uncertainty set Δ is convex, a saddle point for $\mathrm{S}(\boldsymbol{w}; \boldsymbol{a}, \boldsymbol{R})$ exists. Therefore, the solution to problem (10) can be obtained as (14) shows. Since $f(\boldsymbol{a}, \boldsymbol{R}) = \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a}$ is a convex function of $(\boldsymbol{a}, \boldsymbol{R})$, line one of (14) is a convex optimization problem, which is computationally tractable. Then the solution to RMVB with only steering vector uncertainty is given by

$$\begin{cases} \boldsymbol{a}_{\mathrm{wc}} = \arg \min_{\boldsymbol{a}\in\Delta_{\boldsymbol{a}}} \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \\ \boldsymbol{w}_* = e^{j\phi} \dfrac{\boldsymbol{R}^{-1}\boldsymbol{a}_{\mathrm{wc}}}{\boldsymbol{a}_{\mathrm{wc}}^H \boldsymbol{R}^{-1} \boldsymbol{a}_{\mathrm{wc}}} \end{cases} \tag{15}$$

where ϕ is an arbitrary phase. The solution and the objective are in the same form as those of Capon's method, except that the nominal steering vector used in Capon's method is replaced by the worst-case steering vector $\boldsymbol{a}_{\mathrm{wc}}$.
In sum, if the uncertainty set is convex, saddle point exists for $\mathrm{S}(\boldsymbol{w}; \boldsymbol{a}, \boldsymbol{R})$, and the max-min problems are reduced to a convex optimization problem; the solution to RMVB is shown as Capon's beamformer with the worst-case steering vector.

In cases, the uncertainty set does not have specific structure we may use a convex polyhedron to approximate the uncertainty set. The polyderon can be expressed as $\Delta_{\boldsymbol{a}} = \mathrm{conv}\left(\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K\}\right)$, where $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K\}$ must include all vertices of the uncertainty set and might include some points inside the uncertainty set.
Since any $\boldsymbol{a} \in \mathrm{conv}\left(\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K\}\right)$ can be written as $\boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_K \boldsymbol{a}_K$, where $\{\lambda_i\}_{i=1}^K$ are the convex combination coefficients satisfying $\sum_i \lambda_i = 1,\ \lambda_i \geq 0$, the problem

$$\boldsymbol{a}_{\mathrm{wc}} = \arg \min_{\boldsymbol{a}\in\mathrm{conv}(\{\boldsymbol{a}_1,\cdots,\boldsymbol{a}_K\})} \boldsymbol{a}^H \boldsymbol{R}^{-1} \boldsymbol{a} \tag{16}$$

is cast as a quadratic programming (QP) to solve for the optimal $\{\lambda_i\}_{i=1}^K$, that is,

$$\min_{\{\lambda_1,\cdots,\lambda_K\}} \left\| \boldsymbol{R}^{-1/2} \left(\sum_{i=1}^K \lambda_i \boldsymbol{a}_i \right) \right\| \tag{17}$$

$$\text{s.t.} \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \tag{18}$$

The above problem can be expressed in matrix form

$$\min_{\boldsymbol{\lambda}} \quad \boldsymbol{\lambda}^T \left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A} \right) \boldsymbol{\lambda} \tag{19}$$

$$\text{s.t.} \quad \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \tag{20}$$

where $\boldsymbol{\lambda} = [\lambda_1, \cdots, \lambda_K]^T \in \mathbb{R}^K$, $\boldsymbol{A} = [\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K] \in \mathbb{C}^{M\times K}$. Note that $\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A} \in \mathbb{H}_+^K$ with $\mathbb{H}_+^K$ denoting the set of all $K \times K$ Hermitian postive semidefinite matrices. Since $\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A} \in \mathbb{H}_+^K$, $\mathrm{Im}\left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A}\right)$ is an antisymmetric matrix and $\boldsymbol{\lambda}^T \mathrm{Im}\left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A}\right) \boldsymbol{\lambda} = 0$. As a result, (19) can be reduced to the following QP so that the imaginary numbers are avoided

$$\begin{aligned} \min_{\boldsymbol{\lambda}} \quad & \boldsymbol{\lambda}^T \mathrm{Re}\left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A}\right) \boldsymbol{\lambda} \\ \text{s.t.} \quad & \sum_i \lambda_i = 1,\ \lambda_i \geq 0 \end{aligned} \tag{21}$$

where $\mathrm{Re}\left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A}\right) \in \mathbb{S}_+^K$ with $\mathbb{S}_+^K$ denoting the set of all $K \times K$ symmetric postive semidefinite matrices. In sum, the solution to RMVB with $\Delta_{\boldsymbol{a}} = \mathrm{conv}\left(\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_K\}\right)$ is given by

$$\begin{cases} \boldsymbol{\lambda}_* = \arg\min_{\boldsymbol{\lambda}} \quad \boldsymbol{\lambda}^T \mathrm{Re}\left(\boldsymbol{A}^H \boldsymbol{R}^{-1} \boldsymbol{A}\right) \boldsymbol{\lambda} \\ \text{s.t.} \quad \sum_i \lambda_i = 1,\ \lambda_i \geq 0,\ i = 1, \cdots, K \\ \boldsymbol{a}_{\mathrm{wc}} = \boldsymbol{A} \boldsymbol{\lambda}_* \\ \boldsymbol{w}_* = e^{j\phi} \dfrac{\boldsymbol{R}^{-1}\boldsymbol{a}_{\mathrm{wc}}}{\boldsymbol{a}_{\mathrm{wc}}^H \boldsymbol{R}^{-1} \boldsymbol{a}_{\mathrm{wc}}} \end{cases} \tag{22}$$

4. Simulations

Numerical examples of RMVB with ellipsoid/spherical uncertainty can be found in [5]–[7]. Here we only present numerical examples of RMVB with non-convex uncertainty, and we will consider signal direction uncertainty, the most practical steering vector uncertainty.

Consider a uniform linear array consisting of $M = 10$ omnidirectional elements spaced half a wavelength apart. The signal of interest impinges upon the array at DOA $\theta_s = 23°$ with signal-to-noise ratio (SNR) $\sigma_s^2/\sigma_n^2 = 10$ dB. Two interferences arrive at $\theta_{i_1} = -10°$ and $\theta_{i_2} = 60°$, with interference-to-noise ratio (INR) $\sigma_{i_1}^2/\sigma_n^2 = \sigma_{i_2}^2/\sigma_n^2 = 30$ dB. The beamformer does not know the actual DOA of the desired signal, but knows a DOA uncertainty set. The DOA uncertainty set consists of $K = 15$ points equally spaced in $[10°, 30°]$, i.e. $\theta_j = 20° + \frac{2j-K-1}{K-1} \cdot 10°,\ j = 1, \cdots, K$. The corresponding uncertainty set of steering vector is $\{\boldsymbol{a}(\theta_j)\}_{j=1}^{K}$.

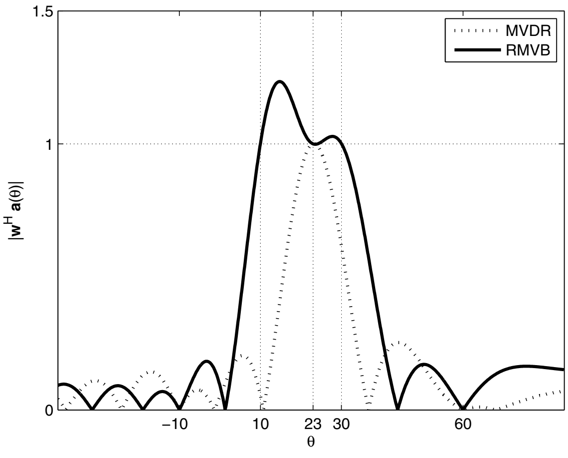

Figure 1. Beampatterns of the RMVB and MVBR beamformers

Figure 1 plots the beampatterns of RMVB beamformer and MVDR beamformer. It can be seen in Fig. 1 that RMVB achieve greater-than-unity gain over the uncertainty set. However, the MVDR beamformer is not able to achieve such gain for most of the points in the uncertainty set. Both the MVDR and RMVB beamformers forms two null towards the direction of the two interference signal. It shall be known that the MVDR beamformer is designed by assuming the signal steering vector is perfectly known.

5. Conclusion

This paper has shown that the robust minimum variance beamforming problem has a simple closed-form solution for convex uncertainty region. It is shown that this solution has a similar form as Capon's beamforming, except that the steering vector in our solution is chosen as the worst-case one in the uncertainty set. Numerical examples have demonstrated the good performance as well as very low computational burden of the new robust beamformer in the presence of signal look direction mismatches.

6. References

[1] J. Capon, "High resolution frequency-wavenumber spectrum analysis," *Proc. IEEE*, vol. 57, no. 8, pp. 1408–1418, Aug. 1969.

[2] H. Cox, R. Zeskind, and M. Owen, "Robust adaptive beamforming," *IEEE Trans. Acoust. Speech, Signal Processing*, vol. 35, pp. 1365–1376, Oct. 1987.

[3] C.-Y. Tseng, and L. J. Griffiths, "An unified approach to the design of linear constraints in minimum variance adaptive beamformers," *IEEE Trans. Antennas Propag.,* vol. 40, pp. 1533–1542, Dec. 1992.

[4] K. L. Bell, Y. Ephraim, and H. L. Van Trees, "A bayesian approach to robust adaptive beamforming," *IEEE Trans. Signal Processing*, vol. 48, no. 2, pp. 386–398, Feb. 2000.

[5] S. A. Vorobyov, A. B. Gershman, and Z.-Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem," *IEEE Trans. Signal Processing*, vol. 51, no. 2, pp. 313–324, Feb. 2003.

[6] J. Li, P. Stoica, and Z. Wang, "On robust Capon beamforming and diagonal loading," *IEEE Trans. Signal Processing*, vol. 51, no. 7, pp. 1702–1715, Jul. 2003.

[7] R. G. Lorenz and S. P. Boyd, "Robust minimum variance beamforming," *IEEE Trans. Signal Processing*, vol. 53, no. 5, pp. 1684–1696, May 2005.

[8] S. Q. Wu, and J. Y. Zhang, "A new robust beamforming method with antennae calibration errors," *Proc. IEEE Wireless Commun. Networking Conf.*, vol. 2, pp. 869–872, Sept. 1999.

[9] S. Verdu and H. Poor, "On minimax robustness: A general approach and applications," *IEEE Trans. Inform. Theory*, vol. 30, no. 2, pp. 328–340, Mar. 1984.

[10] S. P. Boyd, and L. Vandenberghe, *Convex Optimization,* Cambridge, U.K.,: Cambridge Univ. Press, 2004.