Realization of a Radial Uniaxial sphere with a multilayer sphere

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Abstract

A multilayer sphere with a fixed external radius and with an arbitrary number of layers, as a possible realization of a Radial Uniaxial (RU) sphere is presented. The well known quasi-static model for the polarizability of the multilayer sphere is considered. The limits in which the multilayer sphere can be considered a RU sphere are discussed. Moreover different techniques are shown for layering the spheres and it is found which of them is convenient in terms of convergence speed.

1. Introduction

The electromagnetic scattering by a multilayer sphere is widely studied in the literature, because of its interest from the point of view of both theory and applications. The first results on the scattering by a multilayer sphere have been proposed in [1]. In this paper, the stratified sphere is considered as a radial inhomogeneous sphere, in a manner very similar to the radial uniaxial sphere, recently studied in the metamaterial literature [2].

In recent years, the Radial Uniaxial (RU) structures have been widely studied in the literature because of their application to the metamaterial design and in particular to the electromagnetic cloaking [3, 4]. The RU sphere is an anisotropic uniaxial sphere with the permittivity and/or the permeability tensors with the optic axis in the radial direction in a spherical reference frame.

The theoretical premise to the analysis of the scattering by anisotropic objects has been presented in [5, 6]. In particular, the scattering problems by anisotropic spheres have been often faced in the literature [7, 8].

The main purpose of the present paper is to understand the limits in which a stratified sphere with a fixed external radius can be considered homogenized in a RU sphere, by varying the number of the internal layers. We take a layered sphere with two alternating materials, i.e., with only two values of the permittivity. A quasi-static analysis of a multilayer sphere and the polarizabilities of different kinds of multilayer spheres have been calculated. The quasi-static model has been compared with the homogenized RU sphere and the convergence limits between the two models have been established. Moreover, the case of lossy materials has been considered and the convergence of the homogenized model to the stratified sphere has been proved in this case, too.

In Section 2, the quasi-static model is established and the polarizability of the stratified sphere is calculated. Moreover, the homogenization procedure is presented. In Section 3, the validation and the study of the convergence of the homogenized RU model by a comparison with the quasi-static one is shown and the convergence criteria are established. Finally, in Section 4, the conclusions are drawn.

2. Homogenization model

Let us consider a multilayer sphere of alternating relative permittivity ε_1 and ε_2, immersed in a vacuum, see Fig. 1. In the following, the permittivity of the cover layer is always called ε_1. The radius of the external sphere is fixed and equal to a_1 and the other internal radii are:

$$a_k = \frac{N-k+1}{N} a_1 \tag{1}$$

Therefore, as a starting point, we consider the case of equidistant layers. The number of layers N is totally arbitrary.

To obtain the dipole moment of this multilayer sphere, the static problem of an incident field $\boldsymbol{E} = E\boldsymbol{z}_0$ must be faced. To solve the problem, we apply the transmission-line method [9], i.e. the Laplace equation $\nabla^2\phi(\boldsymbol{r}) = 0$ has to be solved in spherical coordinates for the electric potential in all the subregion: $\phi(\boldsymbol{r}) = (-C_i r + D_i r^{-2})\cos\vartheta$. Applying the boundary conditions on all the layers, we obtain the following relation [11]:

$$\binom{C_0}{D_0} = \prod_{i=0}^{N-1} \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} \binom{C_N}{0} \tag{2}$$

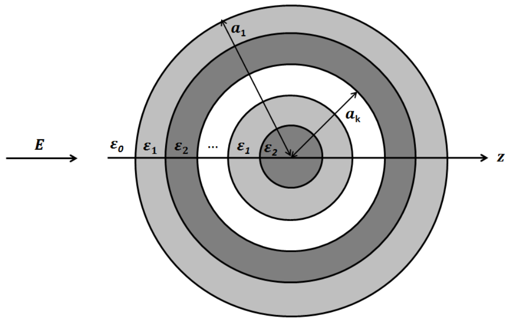

Fig. 1. Geometry of the problem.

where, ε_0 is the vacuum's permittivity, and V is the volume of the external sphere, i.e., $V = \frac{4}{3}\pi a_1^3$. From the knowledge of the polarizability, it is easy to obtain an expression of the effective permittivity of the multilayer sphere [10]:

$$\varepsilon_{eff} = \varepsilon_0 + \frac{\alpha/V}{1-3\varepsilon_0\alpha/V} \tag{4}$$

The quasi-static model allows us to consider the multilayer sphere as an isotropic sphere with the permittivity in (4).
A different model of the multilayer sphere is the RU sphere. In the literature, a layered sphere with two permittivities ε_1 and ε_2 has been considered as a homogeneous sphere with a tensorial permittivity in a spherical reference frame, as follows [11]: $\bar{\bar{\varepsilon}} = \varepsilon_r \boldsymbol{u}_r \boldsymbol{u}_r + \varepsilon_\tau \bar{\bar{I}}_\tau$, where: $\boldsymbol{u}_r$ is the unit vector in the radial direction, $\bar{\bar{I}}_\tau = \bar{\bar{I}} - \boldsymbol{u}_r \boldsymbol{u}_r$ is the tangential unit tensor, and $\bar{\bar{I}}$ is the unit tensor. The quantities ε_r and ε_τ are the relative permittivities in the radial and tangential directions, respectively. The expressions of the radial and tangential permittivities, found in the literature, as a function of the permittivities of the layers in a multilayer sphere are the following [12]:

$$\varepsilon_r = \frac{2}{\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}}, \quad \varepsilon_\tau = \frac{\varepsilon_1+\varepsilon_2}{2} \tag{5}$$

Moreover, the RU sphere can be represented with a polarizability and with an effective permittivity as follows [11]:

$$\alpha_{RU} = 3\frac{\varepsilon_{RU}-1}{\varepsilon_{RU}+2}, \qquad \varepsilon_{RU} = \frac{\varepsilon_r}{2}\left(-1 + \sqrt{1+8\frac{\varepsilon_\tau}{\varepsilon_r}}\right) \tag{6}$$

In the next sections, we are going to investigate what are the limits in which a multilayer sphere can be considered a RU sphere.

3. Validation and Results of the Models

In the present Section, as a first result, we want to show five values of polarizability (Fig. 2). In all the following cases, relatively to Figure 2.a, the values of the permittivity are $\varepsilon = \{2,4\}$: two concentric spheres, with $\varepsilon_1 = 2$ and $\varepsilon_2 = 4$ and viceversa, i.e., $\varepsilon_1 = 4$ and $\varepsilon_2 = 2$; a RU sphere, computed with the formula (6); a multilayer sphere, as a function of the number of the layers, when the cover layer has a permittivity $\varepsilon_1 = 2$ and $\varepsilon_2 = 4$. The radius of the sphere is considered of $a_1 = \lambda_0/10^3$ to respect the quasi-static approximation rule, with λ_0 the wavelength in the air. From Fig. 2.a, we can see that the value of the permittivity of the external sphere has a great weight on the polarizability. In fact, a homogeneous sphere with $\varepsilon = 2$ or $\varepsilon = 4$ has a polarizability $\alpha = 0.75$ or $\alpha = 1.5$, respectively. Both the cases of two concentric spheres are bounded between these two values: in the case with the external sphere with permittivity $\varepsilon_1 = 2$ the polarizability is $\alpha = 0.85$, and in the case with $\varepsilon_1 = 4$ it is $\alpha = 1.42$. On the other hand, the polarizability of the RU sphere takes a value $\alpha = 1.16$, see Eq. (6), that is an intermediate value between the previous two cases. Considering now the case of the multilayer sphere, we can see that if $\varepsilon_1 = 2$, the polarizability starts from the value $\alpha = 0.85$ and suddenly increases to the polarizability of the RU sphere. On the other hand, if we consider a multilayer sphere with an external layer with $\varepsilon_1 = 4$, the polarizability starts from the value $\alpha = 1.42$ and suddenly decreases to the polarizability of the RU sphere. As a consequence, the error in considering a multilayer sphere as a RU sphere decreases when the number of layers grows. The polarizability of the multilayer sphere approaches the polarizability of the RU sphere from greater or lower values depending on the permittivity of the external layer. The same conclusions can be drawn when the permittivities of the multilayer sphere are complex. As it is well known, the media taken into account in the fabrication of metamaterials, often present dielectric losses. In Fig. 2.b, it can be seen that the real and imaginary parts of the polarizability of the multilayer sphere both converge to the real and imaginary parts of the polarizability of the RU sphere, respectively.

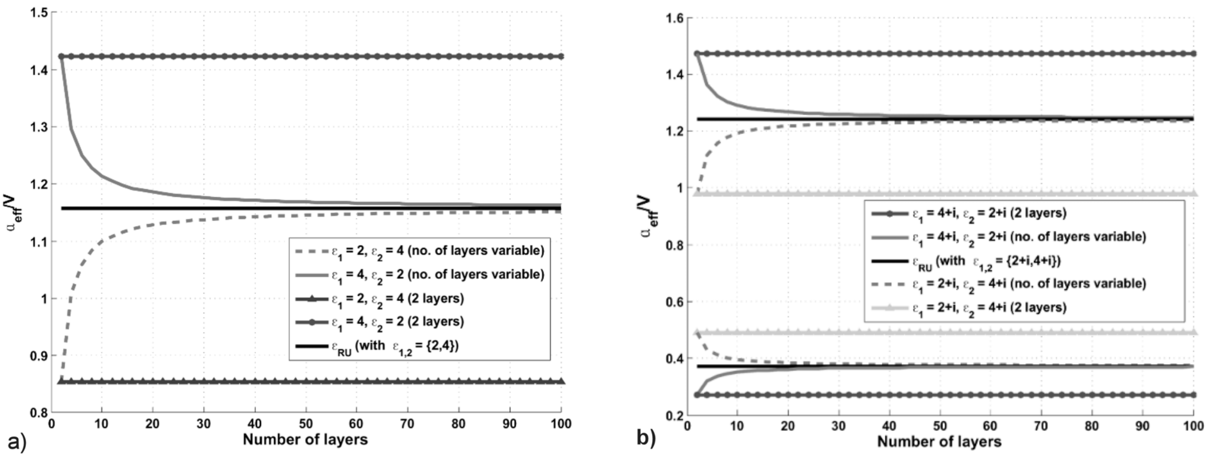

Fig. 2. Normalized polarizability as a function of the number of layers, with the following permittivities: a) $\boldsymbol{\varepsilon} = \{2, 4\}$, b) $\boldsymbol{\varepsilon} = \{2 + i, 4 + i\}$.

The differences between the polarizabilities of the multilayer sphere and the RU sphere are plotted in Fig. 3.a. As can be seen, the difference between the polarizability of the multilayer sphere and that of the RU sphere does not depend on the order of the layer, i.e., it does not depend on the choice of the cover layer.

Until now, we considered a multilayer sphere with equidistant layers, i.e., with the radius of each sphere computed with the formula (1). Obviously, this is not the only way to build a multilayer sphere. We considered other two possibilities: one of them is the case in which each layer has the same volume, i.e., each layer has the volume of the most internal sphere and the other possibility is the same-area rule, i.e. we consider that each sphere has an increase of surface equal to the previous sphere, in formula: $a_k = \left[\frac{N-(k-1)}{N}\right]^{1/3} a_1,\ \ a_k = \left[\frac{N-(k-1)}{N}\right]^{1/2} a_1$, respectively.

In Fig. 3.b, the difference between the effective permittivity of a RU sphere and that of a multilayer sphere as a function of the number of layers, with $\varepsilon_1 = 2$ and $\varepsilon_2 = 4$, is shown with the three different rules. As can be seen, the speed of the convergence varies with the rule type, i.e., the convergence with the same-area rule is quicker than with the same distance rule and the same-volume rule is quicker than the same-area rule.

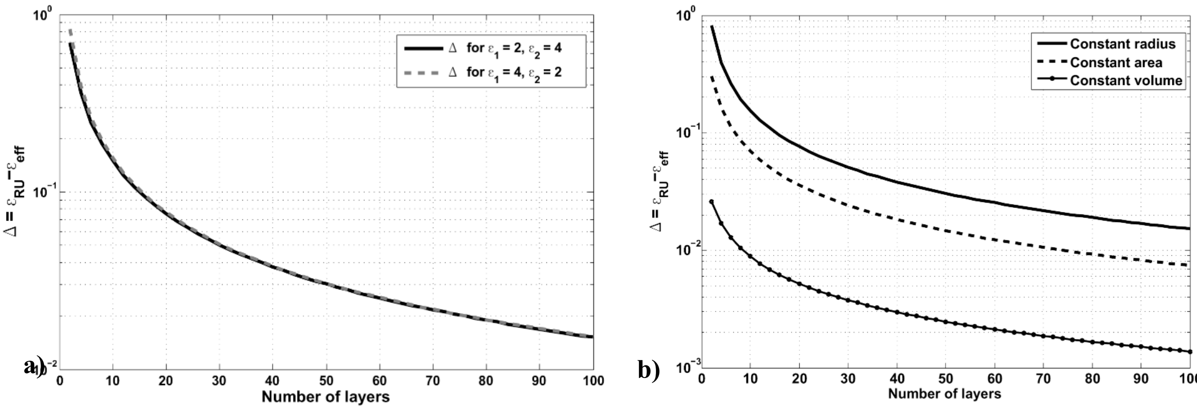

Fig. 3. Difference between the effective permittivity of a RU anisotropic sphere and that of a multilayer sphere as a function of the number of layers with $\boldsymbol{\varepsilon} = \{2, 4\}$; a) in the case of same-distance rule for $\boldsymbol{\varepsilon_1 = 2}$, $\boldsymbol{\varepsilon_2 = 4}$ (solid black line) and for $\boldsymbol{\varepsilon_1 = 4}$, $\boldsymbol{\varepsilon_2 = 2}$ (dashed gray line); b) in three different cases of $\boldsymbol{\varepsilon_1 = 2}$, $\boldsymbol{\varepsilon_2 = 4}$: (solid line) with the same-distance rule, (dashed line) with the same-area rule, and (solid-dotted line) with the same-volume rule.

Another interesting result concerns the number of layers needed to consider the multilayer sphere as a RU sphere for high values of the permittivities. In fact, in many applications high values of the permittivity are needed. In Fig. 4, the number of layers needed to have a difference between the permittivity of the layer and of the RU sphere equal to 1% is plotted as a function of the ratio $\varepsilon_1/\varepsilon_2$. As can be seen, the number of layers grows as a logarithmic function with the permittivity of the cover layer. In fact, the number of layers needed grows exactly as $N = 4\,log^2(\varepsilon_1/\varepsilon_2)$ as can be seen in Fig. 4. This empirical rule can be very useful in the design of a multilayer sphere in order to control the error of the homogenization process.

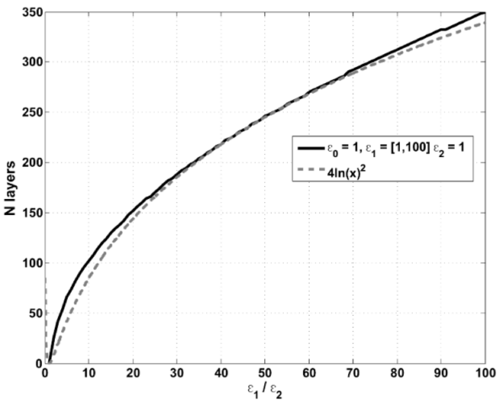

Fig. 4. Number of layers needed to have a difference between the polarizability of the layer and RU sphere equal to 1%, plotted as a function of the ratio $\varepsilon_1/\varepsilon_2$.

4. Conclusion

In this paper, we presented a quasi-static model for the analysis of the electric potential in the presence of a multilayer sphere with an arbitrary number of layers with two alternating values of permittivity. We found closed form expressions for the polarizability of the multilayer sphere and, consequently, for the effective permittivity. Moreover, we analyzed the convergence of the polarizability of the multilayer sphere to the expected value of the effective polarizability of a Radial Uniaxial (RU) sphere. We demonstrated how the polarizability of a multilayer sphere suddenly tends to the polarizability of a RU sphere when increasing the number of layers. Furthermore, we considered three different ways to choose the distance between the layers, i.e., the same-distance, the same-area and the same-volume rules, and we checked the speed of convergence of the three methods increasing the number of layers, finding in the same-volume rule the best choice.

5. References

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