Generalization of Matrix Pencil Methods to the Synthesis of Wideband Aperiodic Linear Arrays with Frequency-Invariant Patterns

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Abstract

The problem of generalizing the matrix pencil methods (MPM) to the synthesis of wideband aperiodic arrays with frequency-invariant (FI) patterns is considered. A new pattern sampling strategy is presented to obtain the multiple pattern data of different frequencies having the common poles. The generalized MPM (GMPM) and its forward-backward version (GFBMPM) are developed to find the best common element positions for different frequencies, which can significantly reduce the number of elements required for the desired pattern shape. Numerical examples are presented to validate the effectiveness and advantages of the proposed methods.

1. Introduction

Wideband antenna arrays have been widely used in many applications such as sonar, communication and radar systems. Among them are a class of arrays with frequency-invariant (FI) patterns which are usually designed for undistorted transmission of wideband signals. These arrays in general have multiple digital filters, each connected to an antenna element, to provide the frequency-dependent excitations required by the desired FI responses. Many practical techniques have been developed to design appropriate filter coefficients for the FI pattern, such as convex optimization methods [1], Fourier transform technique [2], the least-squares approach [3]. However, they are not easily extended to optimize the element positions, and consequently most of them prefix the elements with the spacing of half a wavelength at the highest frequency and sometimes requires a large number of elements to obtain a satisfactory pattern performance.

It is well understood that using nonuniform spacings can reduce the number of elements required for the pattern synthesis. Many advanced techniques have been proposed for the synthesis of nonuniformly spaced arrays [4-7]. Among them, many are proposed for the use at a single frequency or in narrow band, and some techniques designed for synthesizing wideband nonuniformly spaced arrays, for example, the method in [7], do not aim at synthesizing frequency-invariant patterns. Previously we presented non-iterative synthesis techniques based on the matrix pencil method (MPM) and its variants to perform sparse synthesis of linear arrays matching reference pencil beams [8], shaped beams [9] or reconfigurable multiple patterns [10]. All these MPM techniques deal also with the single-frequency synthesis problems. In this paper, the MPM is generalized to be capable of processing multiple sequences, each with different length. The generalized MPM is then applied to synthesize wideband aperiodic arrays with FI patterns. Numerical synthesis examples are presented to validate the effectiveness and advantages of the proposed methods.

2. Frequency Invariant Pattern Synthesis Problem

Consider a wideband linear array with M nonuniformly spaced elements. The wideband array factor is given by

$$P(f, \cos\theta) = \sum_{m=1}^{M} R_m(f) e^{j2\pi f x_m \cos\theta / c} \tag{1}$$

where c is the wave propagation velocity, x_m is the location of the mth element, and $R_m(f)$ is its frequency-dependent complex excitation. Assume that the frequency range of interest is denoted by (f_L, f_U). We set the tested frequencies with $f_k = f_L + (k-1)\Delta_f$ for $k = 1, 2, \cdots, K$, where $\Delta_f = (f_U - f_L)/(K-1)$. Thus the FI pattern synthesis problem is to find the best solution to the following equation with as few elements as possible

$$P_d(\cos\theta) = \sum_{m=1}^{M} R_m(f_k) e^{j\beta_k x_m \cos\theta} \tag{2}$$

where $\beta_k = 2\pi f_k / c$ is the wavenumber corresponding to the kth frequency. Let $\mu_k = \beta_k \cos\theta$ and substitute it into (2), we have $F^{(k)}(\mu_k) = \sum_{m=1}^{M} R_m^{(k)} e^{j x_m \mu_k}$, where $F^{(k)}(\mu_k) = P_d(\mu_k/\beta_k)$ and $R_m^{(k)} = R_m(f_k)$. Since μ_k is the product of the wavenumber β_k and the spatial variable $\cos(\theta)$, it varies within the range $(-\beta_k, \beta_k)$ as θ changes from 0 to π.

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To apply the matrix pencil method to the above problem, we need to sample all the μ_k with the same interval so as to obtain unchanged poles for different frequencies. We choose $u_k = n_k\Delta_\mu$, where $n_k = -N_k/2, -N_k/2+1, \cdots, N_k/2$ for $k = 1, 2, \cdots, K$. According to the Nyquist sampling theorem, the condition that $\Delta_\mu \leq \pi/(2x_{\max})$ must be satisfied, where $x_{\max} = \max\{x_m\}$. Set $N_1 = 2 \cdot \text{round}[\beta_1/\Delta_\mu]$, and $N_k = 2 \cdot \text{floor}[\beta_k N_1/2\beta_1]$ for $2 \leq k \leq K$. Then we have

$$f^{(k)}[n_k] = F^{(k)}(n_k\Delta_\mu) = \sum_{m=1}^{M} I_m^{(k)} z_m^{n_k} \tag{3}$$

where $z_m = e^{jx_m\Delta_\mu}$. As can be seen, the pattern data over different frequencies have different lengths, but all with the same poles.

3. Generalized MPM/FBMPM-based Wideband Array Synthesis

Further generalization of the original MPM is required to deal with the multiple pattern sequences with different lengths. The Generalized MPM (GMPM) organizes all the pattern data into the form of a block Hankel matrix:

$$\mathbf{Y}^{GMP} = \left[\ [\mathbf{Y}^{(1)}]^T,\quad [\mathbf{Y}^{(2)}]^T,\quad [\mathbf{Y}^{(1)}]^T,\quad \cdots, [\mathbf{Y}^{(K)}]^T\ \right]^T \tag{4}$$

$$\mathbf{Y}^{(k)} = \left[\ \mathbf{y}_0^{(k)},\quad \mathbf{y}_1^{(k)},\quad \cdots,\quad \mathbf{y}_L^{(k)}\ \right] \tag{5}$$

where $\mathbf{y}_l^{(k)} = [y_l^{(k)}, y_{l+1}^{(k)}, \cdots, y_{N_k-L+l}^{(k)}]^T$ and $y_l^{(k)} = f^{(k)}[l - N_k/2] = F^{(k)}(l\Delta - N_k\Delta/2)$. It can be seen that the submatrices $\mathbf{Y}^{(k)} \in C^{(N_k-L+1)\times(L+1)}$ are Hankel matrices where the number of rows among the submatrices can be much different, and the whole matrix $\mathbf{Y}^{GMP} \in C^{N_S\times(L+1)}$ where $N_S = \sum_{k=1}^{K} N_k - KL + K$. Then, consider the two matrices $\mathbf{Y}_f^{GMP}$ and $\mathbf{Y}_l^{GMP}$ which are obtained from $\mathbf{Y}^{GMP}$ by deleting the first column and the last column, respectively. If M is the number of elements for a satisfactory FI pattern synthesis, then $\mathbf{Y}_f^{GMP}$ and $\mathbf{Y}_l^{GMP}$ can be respectively factorized as follows.

$$\mathbf{Y}_f^{GMP} = \mathbf{P}_{N_S\times(KM)}\mathbf{A}_{(KM)\times M}\mathbf{Z}_{M\times M}\mathbf{S}_{M\times L} \tag{6}$$

$$\mathbf{Y}_l^{GMP} = \mathbf{P}_{N_S\times(KM)}\mathbf{A}_{(KM)\times M}\mathbf{I}_{M\times M}\mathbf{S}_{M\times L} \tag{7}$$

where $\mathbf{I}_{M\times M}$ is an identity matrix, and the others are shown in the following,

$$\mathbf{P}_{N_S\times(KM)} = \begin{bmatrix} [\mathbf{P}_1]_{(N_1-L+1)\times M} & & & O \\ & [\mathbf{P}_2]_{(N_2-L+1)\times M} & & \\ & & \ddots & \\ O & & & [\mathbf{P}_K]_{(N_K-L+1)\times M} \end{bmatrix} \tag{8}$$

$$[\mathbf{P}_k]_{(N_k-L+1)\times M} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_M \\ \vdots & \vdots & & \vdots \\ z_1^{N_k-L} & z_2^{N_k-L} & \cdots & z_M^{N_k-L} \end{bmatrix} \tag{9}$$

$$[\mathbf{A}]_{(KM)\times M} = \left[\ [\mathbf{A}_1]_{M\times M},\quad [\mathbf{A}_2]_{M\times M},\quad \cdots,\quad [\mathbf{A}_K]_{M\times M}\ \right]^T \tag{10}$$

$$[A_k]_{M\times M} = \text{diag}[R_1^{(k)} z_1^{-N_k/2}, R_2^{(k)} z_2^{-N_k/2}, \cdots, R_M^{(k)} z_M^{-N_k/2}] \tag{11}$$

$$[\mathbf{Z}]_{M\times M} = \text{diag}[z_1, z_2, \cdots, z_M] \tag{12}$$

$$[\mathbf{S}]_{M\times L} = \begin{bmatrix} 1 & z_1 & \cdots & z_1^{L-1} \\ 1 & z_2 & \cdots & z_2^{L-1} \\ \vdots & \vdots & & \vdots \\ 1 & z_M & \cdots & z_M^{L-1} \end{bmatrix} \tag{13}$$

From (6) and (7), it can be forwardly concluded that the poles of z_m ($m = 1, 2, \cdots, M$) are the generalized eigenvalues of the following matrix pencil:

$$\mathbf{Y}_f^{GMP} - z\mathbf{Y}_l^{GMP} \tag{14}$$

Once z_ms are obtained, the element positions can be calculated by $x_m = \ln z_m / j\Delta_\mu$. Obviously, if $|z_m| \neq 1$, then the calculated x_m will be a complex number with a nonphysical imaginary part [9]. The forward-backward (FB) version of the GMPM is developed to overcome this problem. The matrix of GFBMPM is given by

$$\mathbf{Y}^{GFB} = \begin{bmatrix} (\mathbf{y}_0^{GMP})^* & (\mathbf{y}_1^{GMP})^* & \cdots & (\mathbf{y}_L^{GMP})^* \\ \mathbf{y}_L^{GMP} & \mathbf{y}_{L-1}^{GMP} & \cdots & \mathbf{y}_0^{GMP} \end{bmatrix} \tag{15}$$

where $\mathbf{y}_l^{GMP} \in C^{N_S \times 1}$ is the l-th column vector of $\mathbf{Y}^{GMP}$. Clearly, z_ms can be obtained by solving the following forward-backward matrix pencil (GFBMP):

$$\mathbf{Y}_f^{GFB} - z\mathbf{Y}_l^{GFB} \tag{16}$$

where $\mathbf{Y}_f^{GFB}$ and $\mathbf{Y}_l^{GFB}$ are obtained from $\mathbf{Y}^{GMP}$ by deleting the first column and the last column, respectively. It can be proven that the GFBMP has a useful constraint on the distribution of the poles. That is, the poles must be obtained as a pair of $\{z_m, (1/z_m)^*\}$.

In the above procedure, we assumed that M is already known. However, this is not the case in practice. Hence, we usually perform the singular value decomposition (SVD) as $\mathbf{Y}^{GFB} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^H$. Then a rough estimate of $\hat{M}$ can be given by [8]

$$\hat{M} = \min\left\{ M; \left| \sqrt{\frac{\sum_{m=M+1}^{N_\sigma} \sigma_m^2}{\sum_{m=1}^{M} \sigma_m^2}} < \epsilon \right. \right\} \tag{17}$$

where σ_m is the mth singular value in $\mathbf{\Sigma}$, $N_\sigma = \min\{2N_s, L+1\}$, and ϵ is a small positive number. Then we can retain only $\hat{M}$ largest singular values for $\mathbf{\Sigma}$, and then combine the corresponding left and right singular vectors to obtain a rank-$\hat{M}$ matrix, say $\hat{\mathbf{Y}}^{GFB}$. This matrix corresponds to a FI pattern that can be realizable by only $\hat{M}$ elements.

Note that once one finds the element positions, the complex excitation of $R_m(f)$ at each frequency f_k can be obtained by the least-squares solution as did in the previous MPM methods of [8]. The $R_m(f)$ can be considered as the system transfer function of each element channel, and can be accomplished by either the analog hardware or digital filtering. It is omitted here due to the page limit.

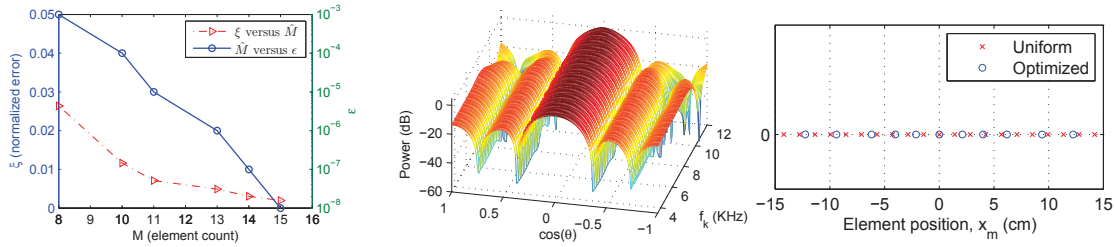
4. Numerical Results

The first example is to synthesize an acoustic array with the desired pattern given by $P_d(\cos\theta) = \sum_{q=-3}^{3} I_q \exp(-jq\pi\cos\theta)$, where I_q ={0.0307, 0.2028, 0.1663, 0.2004, 0.1663, 0.2028, 0.0307}. This pattern was synthesized in [2] by using 21 acoustic sensors with an uniform spacing of 1.42cm. The frequency range achievable for invariant pattern is about 3.5KHz∼12KHz, as shown in Fig. 7 of [2]. We set K=32 tested frequencies uniformly distributed in this band. Choose $\Delta_\mu = \pi/(4x_{\max})$ where $x_{\max} = 0.142$, then set $N_1 = 24$ and $N_k = 2 \cdot \text{floor}[\beta_k N_1/2\beta_1]$ for $2 \leq k \leq 32$. Fig. 1(a) shows the estimated $\hat{M}$ versus $\epsilon \in [10^{-8}, 10^{-3}]$ for the case of $L = 24$. The error ξ versus different $\hat{M}$ is also shown in this figure. We can see that the estimated $\hat{M}$ increases from 8 to 15 as ϵ decreases, and the error ξ decreases with increasing $\hat{M}$ as expected. The synthesized wideband frequency-invariant pattern by using 11 nonuniformly spaced elements is already acceptable, as shown in Fig. 1(b). This pattern remains invariant, at the least for the mainlobe, over the frequency range of 3.5KHz∼12KHz. The 11 element positions optimized by the GFBMPM is shown in Fig. 1(c). For comparison, the 21 uniform element positions used in [2] is also plotted. In this case, we saved 47.6% elements, compared with the uniformly spaced array of [2].

The second example is given for synthesizing a pattern given by Taylor's linear source with the extension $x_{\max} = 16\lambda_U$, $SLL = -40$ dB and $\bar{n} = 7$, depicted by the thick line of Fig.2(a). Assume that the frequency range of interest is between $f_L = 0.5$ GHz and $f_U = 1$ GHz. We set $K = 16$, $\Delta_\mu = \pi/(3x_{\max})$, $N_1 = 2 \cdot \text{round}[\beta_1/\Delta_\mu] = 96$. For this low SLL case, $\epsilon = 10^{-6}$ is used for the criterion of (17), which gives an estimate of $\hat{M} = 38$. For this example, the pencil parameter $L = N_1 - 2$, gives the minimum error of $\xi = 7.29 \times 10^{-5}$. Fig. 2(a) shows the synthesized wideband pattern at each specified frequency, and Fig. 2(b) shows this pattern in the space-frequency plane. As can be seen, although some of sidelobe levels in the synthesized pattern are increased a little bit at high frequency, but they are still less than -40 dB. The synthesized 38 element positions are nonuniformly but symmetrically distributed in ($\pm 15.68\lambda_U$). Compared with the array by using 63 $\lambda_U/2$-spaced elements to occupy the same aperture, we saved 39.7% antenna elements.

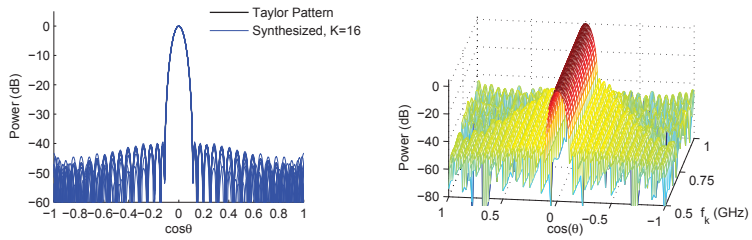
5. Conclusion

We have presented the generalized FBMPM-based synthesis method to design sparse aperiodic arrays with frequency-invariant patterns. Significant element savings have been achieved for the tested cases. The synthesis results also confirm that the wideband FI tylor's pattern with very low sidelobe levels can be obtained by using sparse arrays with optimized element positions and excitations. The synthesized minimum spacing is usually much larger than $\lambda_U/2$, which leaves more space for wideband element design.



(a) (b) (c)

Figure 1: Synthesis of a wideband acoustic array. (a) the estimated $\hat{M}$ versus ϵ (with $L = N_1 = 24$), and in result the error ξ versus $\hat{M}$; (b) the synthesized frequency-invariant pattern with $\hat{M} = 11$; (c) the synthesized element positions and the 21 uniform positions used in [2].



(a) (b)

Figure 2: The synthesized wideband tylor's pattern with the parameters $K = 16$, $N_1 = 96$, $L = 94$ and $\hat{M} = 38$.

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