Scattering and Diffraction of an Arbitrarily Directed Complex-Source Beam by a Semi-Infinite Circular Cone

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Abstract

The problem of an arbitrarily directed Gaussian beam (GB) illuminating an acoustically soft or hard semi-infinite circular cone is solved by using a complex source beam (CSB) whose waist and direction are defined by the real and imaginary parts of the source coordinate, respectively. The corresponding scalar boundary-value problem is solved by assigning a complex-valued source coordinate into the conventional spherical-multipole expression of the Green's function, thus converting it to the response to the incident CSB. The solution requires the calculation of the associated Legendre functions of the 1st kind for a complex-valued argument. Beside a numerical analysis of these calculations, we also present numerical results for the total near- and far-fields.

1 Formulation of the problem and motivation

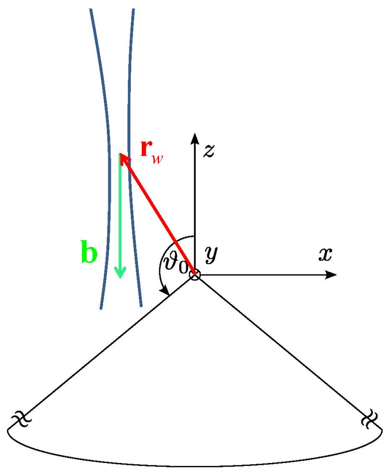

Figure 1: Semi-infinite cone illuminated by an arbitrarily directed complex-source beam.

Consider an acoustically soft (index s) or hard (h) semi-infinite circular cone lying around the negative z-axis as depicted in Fig. 1. In spherical coordinates r, ϑ, φ with the z-axis as polar axis the surface of the cone is given by $\vartheta = \vartheta_0$. The cone is illuminated by a complex-source beam (CSB) with its waist being located at $\boldsymbol{r}_w$ and which is travelling in the direction of $\boldsymbol{b}$. We are looking for the total (diffracted) field.

The combination of the CSB technique and the eigenvalue analysis allows to probe a specific part of the semi-infinite cone and hence to isolate the fields scattered by any part of the cone - including the tip. The long-term goal of this analysis is to incorporate such isolated field parts in the context of asymptotic methods like the Uniform Theory of Diffraction (UTD) [1] and the Complex-Ray Technique [2].

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2 Solution of the Helmholtz equation in the presence of an acoustically soft or hard semi-infinite cone

For a point source located at $\boldsymbol{r}_c$, the normalized total field in the presence of an acoustically soft (index s) and hard (index h) cone, respectively, consists of a linear superposition of all elementary solutions of the Helmholtz equation $\Delta\Phi(\boldsymbol{r}) + \kappa^2\Phi(\boldsymbol{r}) = 0$ in spherical coordinates r, ϑ, φ (with the z-axis as polar axis) and can be represented by means of the symmetric bilinear expansion [3]

$$\Phi_\tau(\boldsymbol{r}) = \sum_{\nu_\tau, m} j_{\nu_\tau}(\kappa r_<) h^{(2)}_{\nu_\tau}(\kappa r_>) Y^m_{\nu_\tau}(\vartheta, \varphi) Y^{m*}_{\nu_\tau}(\vartheta_c, \varphi_c) \tag{1}$$

where $(r_< = r, r_> = r_c)$ if $|\boldsymbol{r}| \leq |\boldsymbol{r}_c|$, $(r_> = r, r_< = r_c)$ if $|\boldsymbol{r}_c| \leq |\boldsymbol{r}|$, $\tau \in \{s, h\}$, and $*$ denotes a complex conjugation. Since the solution has to be regular everywhere and to satisfy the radiation condition, spherical Bessel functions of the 1^{st} kind ($j_\nu(\kappa r_<)$) and spherical Hankel functions of the 2^{nd} kind ($h^{(2)}_\nu(\kappa r_>)$) are employed at a time factor $e^{+j\omega t}$. The surface spherical harmonics Y^m_ν are represented by

$$Y^m_{\nu_\tau}(\vartheta, \varphi) = N^m_{\nu_\tau} P^m_{\nu_\tau}(\cos\vartheta) e^{jm\varphi} \tag{2}$$

where $N^m_{\nu_\tau}$ is a suitable normalization factor and $P^m_{\nu_\tau}(\cos\vartheta)$ represents an associated Legendre function of the 1^{st} kind of order ν_τ and degree m. To satisfy the appropriate boundary condition at $\vartheta = \vartheta_0$ the eigenvalues ν_s and ν_h are chosen according to

$$P^m_{\nu_s}(\cos\vartheta_0) = 0 \quad ; \quad \left.\frac{\partial P^m_{\nu_h}(\cos\vartheta)}{\partial\vartheta}\right|_{\vartheta=\vartheta_0} = 0 \tag{3}$$

for an acoustically soft (Dirichlet case) and hard (Neumann case) cone, respectively. Because the solutions are 2π periodic in φ, the m have to be integers ($m = 0, \pm 1, \pm 2, \dots$). Moreover, as shown in [5] it holds that $m^2 \leq \nu(\nu+1)$. Consequently, there are discrete pairs of eigenvalues (ν_s, m) for the soft cone and (ν_h, m) for the hard cone, respectively, as represented in Fig. 2. Because of the behavior of spherical Bessel functions of the 1^{st} kind, for the given problem we need all eigenvalue pairs up to a chosen maximum order ν^{max}_τ.

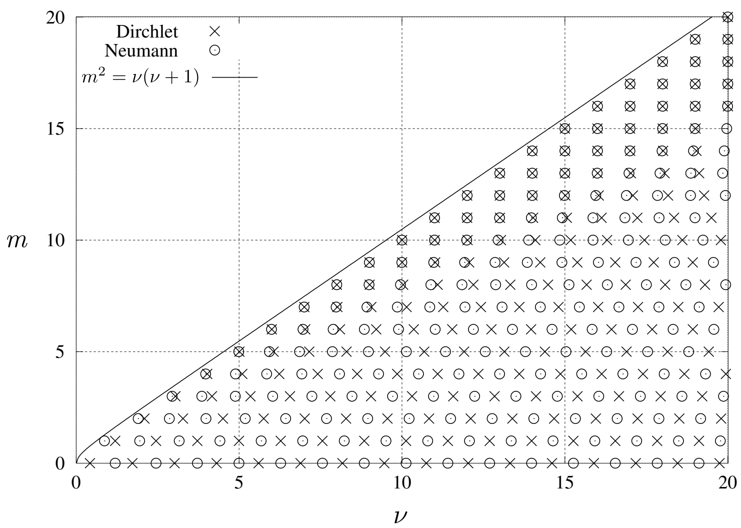

Figure 2: Dirichlet (soft cone) and Neumann (hard cone) eigenvalue pairs for a cone with $\vartheta_0 = 140°$; $\nu^{max}_\tau = 20$; $m > 0$.

3 Evaluation of an arbitrarily directed complex-source beam

To form a complex-source beam (CSB), a complex-valued location $\boldsymbol{r}_c$ is assigned to a point source according to

$$\boldsymbol{r}_c = \boldsymbol{r}_w - j\boldsymbol{b} \quad ; \quad (\boldsymbol{r}_w, \boldsymbol{b} \in \mathbf{R}). \tag{4}$$

Here, $\boldsymbol{r}_w$ is the location of the beam's waist, $\boldsymbol{b}/|\boldsymbol{b}|$ represents the beam's direction, and $|\boldsymbol{b}|$ determines its collimation (or Rayleigh) length. It can be shown, that in a paraxial approximation the CSB describes a Gaussian beam [4]. For a beam directed towards the origin, i.e. the cone's tip, it has been shown in [5] that the spherical components of $\boldsymbol{r}_c = \boldsymbol{r}_w - j\boldsymbol{b}$ are found as

$$
\begin{aligned}
r_c &= r_w + jb &(5)\\
\vartheta_c &= \vartheta_w &(6)\\
\varphi_c &= \varphi_w , &(7)
\end{aligned}
$$

that is, ϑ_c and φ_c remain real-valued. However, for an arbitrarily directed beam we have complex values ϑ_c and φ_c in the multipole expansion (1). The necessity to efficiently evaluate the associated Legendre functions of the first kind for a complex-valued argument ϑ_c and a real-valued order ν turns out to be a delicate task. The representation via Ferrer's function [6]

$$
P_\nu^m(x) = (-1)^m \frac{\Gamma(\nu+m+1)}{2^m\,\Gamma(\nu-m+1)}(1-x^2)^{m/2}\frac{1}{\Gamma(m+1)}\,{}_2F_1(\nu+m+1, m-\nu; m+1; \tfrac{1}{2}-\tfrac{1}{2}x)\,. \tag{8}
$$

was identified as a promising approach. Here the Gaussian hypergeometric function ${}_2F_1$ is defined as

$$
{}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k}\,\frac{z^k}{k!}\,. \tag{9}
$$

where $(a)_k = a(a+1)(a+2)\dots(a+k-1)$ is the Pochhammer symbol.

4 Numerical Results

Figure 3 shows a comparison between a computation of $|P_\nu^m(x)|$ by the hypergeometric representation and a purely numerical routine (quadrature) as well as the corresponding relative error, both as a function of the real part for a constant imaginary part of x. Figure 4 a) represents the diffracted (total) field for an incident CSB with a waist at $\boldsymbol{r}_w/\lambda = 3\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{z}}$ (right side to the tip, travelling in the $-\hat{\boldsymbol{x}}$-direction. We clearly observe the reflected field and the shadowing effect of the cone. Finally, in Figure 4 b) we see the corresponding total radiated far-field, i.e., the scattered field plus the outwardly traveling part of the incident (CSB) field.

5 Acknowledgements

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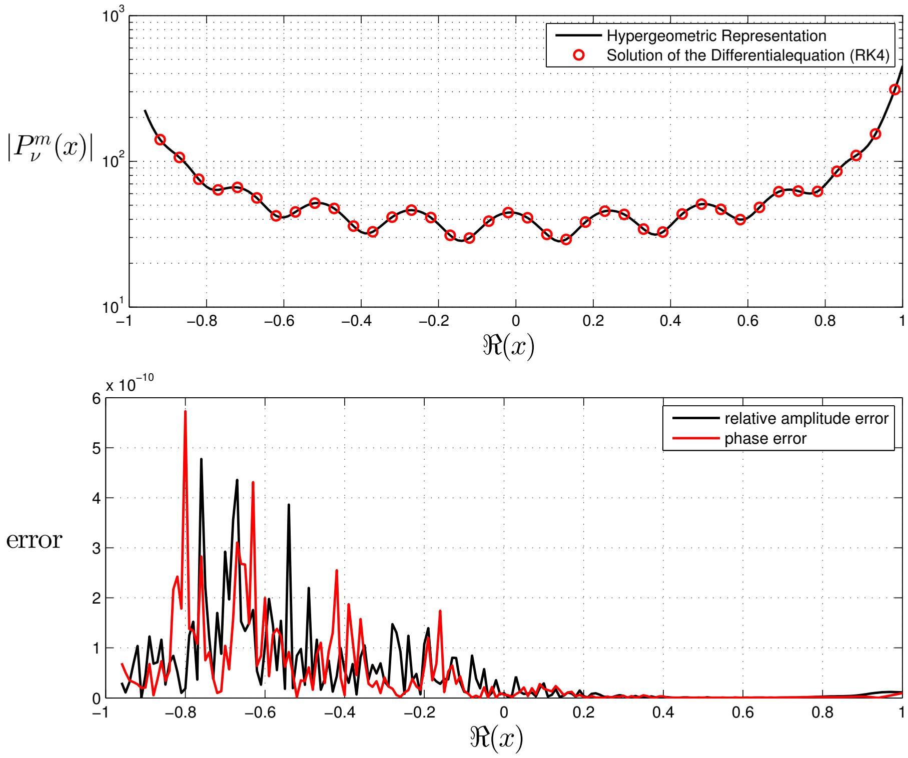

Figure 3: Comparison between computations by the hypergeometric representation and by a purely numerical routine (quadrature) of $|P_\nu^m(x)|$. Relative difference of $|P_\nu^m(x)|$ and absolute difference of $\arg\{P_\nu^m(x)\}$ as a function of $\Re\{x\}$ for a constant imaginary part of the argument $\Im\{x)\} = 0.0602$; $\nu = 11.89378, m = 2$.

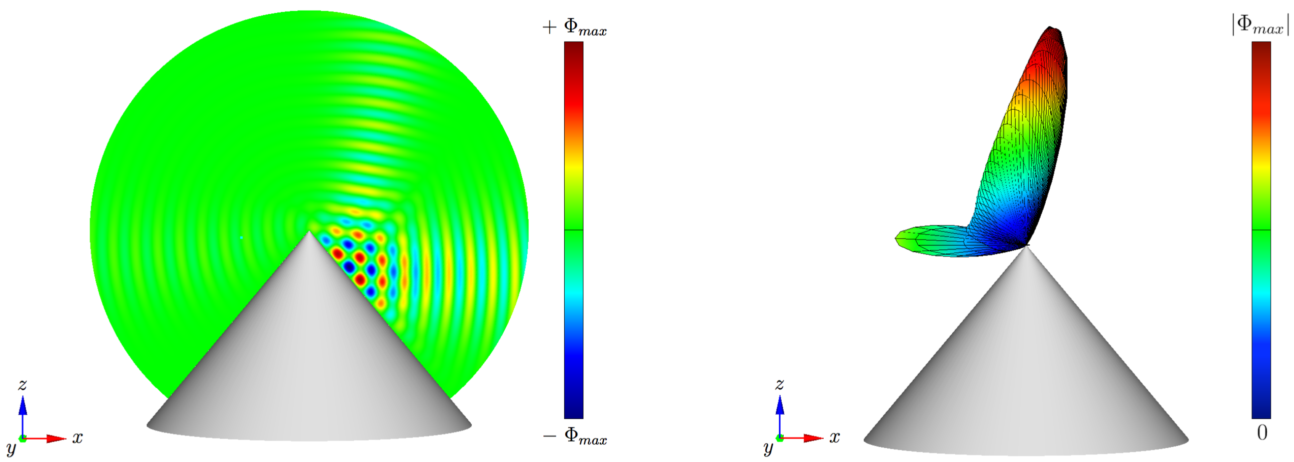

Figure 4: Total near field in the region $r < 10\lambda$ (left) and radiation pattern of the total radiated far field (right). The complex source coordinates are $\boldsymbol{r}_c/\lambda = (3\hat{\boldsymbol{x}} - 2\hat{\boldsymbol{z}}) - j(-7\hat{\boldsymbol{x}})$; $\vartheta_0 = 140°$.