Mathematical Aspects of the Theory of Wave Propagation in Metal–Dielectric Waveguides

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Abstract

We consider fundamental issues of the mathematical theory of the wave propagation in waveguides with inclusions. Analysis is performed in terms of a boundary eigenvalue problem for the Maxwell equations which is reduced to an eigenvalue problem for an operator pencil. We prove that the spectrum of normal waves forms a nonempty set of isolated points localized in a strip with at most finitely many real points. We show the importance of these results for the theory of wave propagation in open guiding structures and consider in more detail the surface wave spectrum of the Goubau line.

1. Introduction

Analysis of the wave propagation in waveguides with nonhomogeneous filling and arbitrary inclusions (perfectly conducting and dielectric) constitutes an important class of vector electromagnetic problems. The fundamental results known for empty waveguides since the late 1940s (existence of normal waves, discreteness and localization of their spectrum on the complex plane, completeness and basis property in terms of both longitudinal and transversal field components, and so on) have been solved [1, 2] for arbitrary shielded metal–dielectric waveguides only recently. Let us briefly summarize these new fundamental results obtained in [1, 2] for an arbitrary metal–dielectric waveguide with nonhomogeneous filling and arbitrary inclusions: the spectrum of normal waves a) is nonempty, b) forms a countable set of isolated points on the complex plane (cut along two intervals on the real axis) without finite accumulation points, and c) is symmetric with respect to the axes on the complex plane, is localized in a strip, contains not more than a finite number of real points, and the spectral points enter the spectrum in 'fours'.

2. Analysis using Spectral Theory of Operator Pencils

Let us give a brief insight into mathematical methods used for obtaining the results formulated in Introduction. The problem on normal waves in a shielded metal–dielectric waveguide filled with nonhomogeneous media and having the boundary with 'edges', stated initially for homogeneous Maxwell equations, is reduced to a nonselfadjoint boundary eigenvalue problem for the system of Helmholtz equations, where the spectral parameter enters the transmission conditions in nonlinear manner; a weak solution is defined and sought using variational relations in Sobolev spaces. Next, the problem is reduced to the study of a fourth-order operator pencil $L(\gamma)$ (neither Keldysh nor hyperbolic). We investigate the properties of the operators of the pencil and establish basic properties of its spectrum listed above. By investigating the system

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of eigenvectors and associated vectors of pencil $L(\gamma)$ it is established, under certain conditions, the double completeness of the system with a finite defect or without a defect using perturbation techniques applied for a pencil of simple structure and factorization of the pencil.

The initial eigenvalue problem on normal waves for Maxwell equations in a shielded metal–dielectric waveguide is reduced in terms of longitudinal electromagnetic field components Π, Ψ to the variational problem [1]

$$\begin{gathered}\gamma^4 \int\limits_{\Omega} (\varepsilon \Pi \bar{u} + \Psi \bar{v})\, dx + \gamma^2 \left(\int\limits_{\Omega} (\varepsilon \nabla \Pi \nabla \bar{u} + \nabla \Psi \nabla \bar{v})\, dx - \right. \\ \left. - (\varepsilon_1 + \varepsilon_2) \int\limits_{\Omega} (\varepsilon \Pi \bar{u} + \Psi \bar{v})\, dx \right) + (\varepsilon_1 - \varepsilon_2)\, \gamma \int\limits_{\Gamma} \left(\tfrac{\partial \Pi}{\partial \tau} \bar{v} - \tfrac{\partial \Psi}{\partial \tau} \bar{u} \right) d\tau + \\ + \varepsilon_1 \varepsilon_2 \left(\int\limits_{\Omega} (\varepsilon \Pi \bar{u} + \Psi \bar{v})\, dx - \int\limits_{\Omega} \left(\nabla \Pi \nabla \bar{u} + \tfrac{1}{\varepsilon} \nabla \Psi \nabla \bar{v} \right) dx \right) = 0, \\ \forall u \in H_0^1(\Omega),\ v \in \hat{H}^1,\end{gathered} \tag{1}$$

which can be written in the operator form as a homogeneous equation for the operator-valued pencil

$$\begin{gathered} L(\gamma) f = 0,\ L(\gamma) : H \to H, \\ L(\gamma) := \gamma^4 K + \gamma^2 (A_1 - (\varepsilon_1 + \varepsilon_2) K) + (\varepsilon_1 - \varepsilon_2)\, \gamma S + \varepsilon_1 \varepsilon_2 (K - A_2), \end{gathered} \tag{2}$$

where all the operators are bounded. Here H_0^1, $\hat{H}^1$, and H denote appropriate Sobolev spaces defined in [1, 2], ε_1 and ε_2 are two different real constants and Γ and Ω denote, respectively, the domain ('bulk') of the waveguide cross-section and the set of interface curves so that S is the 'interface' operator. Eigenvalues and eigenvectors of the pencil coincide with eigenvalues and eigenfunctions of the initial eigenvalue problem on normal waves for Maxwell equations for $\gamma^2 \neq \varepsilon_1$, $\gamma^2 \neq \varepsilon_2$. Operators in (1) possess [1] the following properties: A_1 and A_2 are uniformly positive: $I \leq A_1 \leq \varepsilon_{\max} I$, $\varepsilon_{\max}^{-1} I \leq A_2 \leq I$, where $\varepsilon_{\max} = \max(\varepsilon_1, \varepsilon_2)$ and I is the unit operator in H. Operator S is selfadjoint, $S = S^*$, and $-\frac{1}{2} I \leq S \leq \frac{1}{2} I$. Operator K is positive, $K > 0$, and compact; the following estimate holds for its eigenvalues $\lambda_n(K) = O\left(n^{-1}\right)$, $n \to \infty$. Next, $L(\gamma)$ is self-adjoint and the spectrum of $L(\gamma)$ lies in the strip Π_l for a certain $l > 0$:

$$\sigma(L) \subset \Pi_l = \{\gamma : |Re\gamma| < l\};$$

and is symmetric with respect to the real and imaginary axes: $\sigma(L) = \overline{\sigma(L)} = -\sigma(L)$. Namely, if γ_0 is an eigenvalue of pencil $L(\gamma)$ corresponding to the eigenvector $(\Pi, \Psi)^{\mathrm{T}}$ then $-\gamma_0, \bar{\gamma}_0$, and $-\bar{\gamma}_0$ are also eigenvalues of pencil $L(\gamma)$ corresponding to the eigenvectors $(-\Pi, \Psi)^{\mathrm{T}}$, $\left(\bar{\Pi}, \bar{\Psi}\right)^{\mathrm{T}}$, and $\left(-\bar{\Pi}, \bar{\Psi}\right)^{\mathrm{T}}$ with the same multiplicity.

Set $\delta = (\varepsilon_2 - \varepsilon_1)/2$ and

$$I_0 = \left\{ \gamma : Im\gamma = 0, \frac{\left(\delta^2 + 4\varepsilon_1\right)^{1/2} - |\delta|}{2} \leq |\gamma| \leq \frac{\left(\delta^2 + 4\varepsilon_2\right)^{1/2} + |\delta|}{2} \right\}.$$

In the domain $\mathbf{C} \backslash I_0$ (a) the spectrum of pencil $\sigma(L)$ is a set of isolated eigenvalues with finite algebraic multiplicity, the points $\gamma_j = \pm\sqrt{\varepsilon_i}$ $(i = 1, 2)$ are the degeneration values of $L(\gamma)$: $\dim \ker L(\gamma_j) = \infty$; (b) the spectrum of $L(\gamma)$ forms a countable set of isolated eigenvalues of finite algebraic multiplicity with the only accumulation point at infinity.

From the physical viewpoint the real and pure imaginary points of spectrum $\sigma(L)$ are of interest because they correspond to propagating and decaying waves. It should be noted however that 'complex' waves may exist for $\gamma_0 \in \sigma(L)$ and $\gamma_0' \cdot \gamma_0'' \neq 0$ $(\gamma_0 = \gamma_0' + i\gamma_0'')$. In the general case, strip Π_l cannot be replaced by the set $\Pi_0 = \{\gamma : (Re\gamma) \cdot (Im\gamma) = 0\}$. Also complex waves

occur in 'fours'. Note also that if a waveguide has a homogeneous filling ($\varepsilon_1 = \varepsilon_2$) then there are no complex waves.

Equation (2) may be used for analytical and numerical determination of the spectrum as a good alternative to all other known methods. In fact, all the operators in (2) are *independent* of the spectral parameter and the order of the involved differential operations does not exceed one which greatly simplifies discretization and computations.

3. Open Metal–Dielectric Waveguides

We see that the spectrum of normal waves of a broad family of shielded metal–dielectric waveguides has a definite structure and location on the complex plane. A mathematically justified regular transition to an open waveguide can be made by cutting a small opening (a slot) in the outer boundary ('shield', a closed curve S) so that the interface acquires an additional part $\Gamma_e \subset S$, adding conditions at infinity, e.g. in the form of partial radiation condition and considering the spectral problem on a complicated multi-sheet complex manifold W [3]. Then the spectral set is preserved and undergoes a regular perturbation (except maybe a finite number of points and all real points may become complex with a small imaginary part). This regular perturbation is described and justified in [4, 5] for cavity-backed screens with narrow slots and cylindrical slot lines. Note that a comprehensive study of the spectrum perturbation in slotted structures with circular symmetry, particularly with respect to the increasing diameter of the slot is performed in [6–8]. Thus the results obtained for shielded waveguides are of big importance for the corresponding perturbed families of open structures. When the size of Γ_e increases and in the end coincides with S, the spectral points may be shifted to infinity or to sheets of W through cuts or branch points.

The simplest basic example of an open metal–dielectric waveguide is the Goubau line [9, 10]: a perfectly conducting tube covered by a concentric layer of dielectric. In this case $\Gamma_e = S$ and it may be possible to reduce the spectrum determination to finding zeros of certain characteristic functions that can be obtained explicitly. The simplest spectral problem here is the determination of surface waves, i.e. azimuthally-independent solutions exponentially decaying at infinity which is reduced to finding the (real and complex) roots of the dispersion equation (DE)

$$\mathcal{G}(\beta) \equiv G_g(x,u) - F_g(x,s) = 0, \tag{3}$$

where

$$\begin{aligned} G_g(x,u) &= \epsilon\sqrt{u^2-x^2}\frac{K_0(s\sqrt{u^2-x^2})}{K_1(s\sqrt{u^2-x^2})}, \quad F_g(x,s) = x\frac{\Phi_1(x,s)}{\Phi_2(x,s)}, \\ \Phi_1(x,s) &= J_0(sx)Y_0(x) - J_0(x)Y_0(sx), \quad \Phi_2(x,s) = J_0(x)Y_1(sx) - J_1(sx)Y_0(x), \end{aligned} \tag{4}$$

$$\beta = \frac{\tilde{\beta}}{k_0}, \;\; x = k_0 a\sqrt{\epsilon-\beta^2}, \;\; u = k_0 a\sqrt{\epsilon-1}, \;\; s = \frac{b}{a} > 1, \tag{5}$$

a and b, $b > a$, are the radii of the internal (perfectly conducting) and external (dielectric) cylinders, $\tilde{\beta}$ is the longitudinal wavenumber (spectral parameter), k_0 is the free-space wavenumber $J_m(x)$, $Y_m(x)$, and $K_m(x)$ denote, respectively, the Bessel, Neumann, and McDonald functions of the order $m = 0, 1$. It can be shown that the DE has one real root for arbitrary positive a. Remarkably, it can be proved [9] using the parameter-differentiation method that under the condition $u > h_1^{10}(1)$, where $h_m^{10}(1)$ is the mth root of the function $\Phi_2(x,1) \equiv J_0(x)Y_1(x) - J_1(x)Y_0(x)$, $m = 1, 2, \ldots$, for $s = b/a > 1$ which is sufficiently close to unity, there exists a root $x_0 = x_0(s)$ of DE (3). In other words, for an arbitrarily thin dielectric layer, there exists

a surface wave propagating in the Goubau line—at least for sufficiently large radius $k_0 a$ of the perfectly conducting cylinder—and these roots are perturbed values of $h_m^{10}(1)$.

Complex spectral points (and the corresponding complex surface waves) are complex zeros of the characteristic function $\mathcal{G}(\beta)$ in (3). Numerical verification and calculation of the complex zeros are reported in several works (see e.g. [10] and bibliography therein). However the rigorous proofs of the existence and distribution of zeros of $\mathcal{G}(\beta)$ on the complex plane are not available, to the best of or knowledge. The corresponding analysis goes beyond the scope of the present work and can be performed in a separate study using the theory of functions of several complex variables.

4. Conclusion

We see how difficult is to describe complex spectrum of normal waves already for the simplest open metal–dielectric waveguide. In view of the available complete description of the complex wave spectrum for a broad family of shielded metal–dielectric waveguides summarized in this work the following conclusion could be made: the use of perturbation techniques, in particular parameter-differentiation method applied to the analysis of implicit functions generated by DEs with a reduction to the Cauchy problem is the most promising tool for successful analysis of the existence and distribution on the complex plane of the spectrum of normal waves for both open and shielded waveguides.

5. References

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