Accuracy of the Mortar Element Electric Field Integral Equation

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Abstract

Scattering of time-harmonic electromagnetic fields by perfect electric conductors can be modelled by the electric field integral equation (EFIE). The EFIE is flexible in that it can be applied to both closed and open structures, and that it can be extended to include the effects of a non-zero surface impedance. Approximate solutions of the EFIE can be computed by application of the boundary element method (BEM). In the BEM, the geometry is approximated by a triangular mesh, and the unknown current is approximated by an expansion in basis functions that are constructed subordinate to that mesh. The classic EFIE requires the candidate solution to have continuous normal components everywhere on the surface. This requires the underlying triangulation to be geometrically conforming. Recently, the mortar element EFIE was introduced. In the mortar element EFIE, the candidate solution is only required to have continuous normal components on subsets of the surface. Global normal continuity is imposed in a weak discreet sense by solution of a saddle point problem. The lack of normal continuity of the solution makes it challenging to assess the accuracy of the mortar element EFIE. In this contribution, an error criterion is designed and applied to the solution of the mortar element EFIE. The accuracy of the solution as a function of the mesh parameter is studied.

1. Introduction

Scattering of time-harmonic electromagnetic fields by perfect electric conductors can be modelled by the electric field integral equation (EFIE). The EFIE can be applied to both closed and open structures. The EFIE is flexible in that it can be applied to both closed and open structures. The EFIE can model the skin effect, i.e. the loss of energy as a consequence of the presence of induced currents in a small layer of finite conductivity near the scatterer's surface.

The EFIE can usually not be solved exactly, but approximate solutions can be constructed by application of the boundary element method (BEM). Application of the BEM entails three major steps:

- Approximation of the surface of the scatterer by a triangular mesh. Care needs to be taken that the triangle size is small enough to allow for an accurate description of both the geometry and the solution.
- Approximation of the unknown current by an expansion in basis functions. The problem of finding the currents density is hereby replaced with the problem of finding the unknown coefficients in this expansion.
- Testing of the integral equation by an appropriate set of testing functions, equal in number to the basis functions. This reduced the integral equation to a square system of linear equations.

In order for the approximate solution to this linear system to be optimal, it is required that the space of candidate solutions spanned by the basis functions to be a subset of the space of current densities radiating a field of finite energy. In practice this requires that the basis functions have continuous normal components everywhere on the surface. This in turn requires that the triangular mesh be geometrically conforming, i.e. that two neighbouring triangles share exactly one common edge.

There are several reasons why geometrical meshes might be too restrictive. (i) It is possible that different subcomponents of the scattering structure have been designed and meshed separately, (ii) it might be advantageous to define non-conforming basis functions near the boundary between different subcomponents in order to allow for rigorous geometric separation during the parallel solution of the scattering problem, or (iii) refinements of the mesh that are introduced to locally improve the accuracy of the solution can result in meshes that are not geometrically conforming.

Recently [1], a mortar element method for the EFIE has been introduced, along the lines of work done before in acoustics [2]. The mortar element EFIE allows the candidate solution to have discontinuous normal components along subdomain boundaries. Continuity of the normal components is enforced in a weak discrete sense by solution of a saddle

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point problem. The resulting solution has normal components that are only approximately continuous. This complicates the analysis of the accuracy of the method. One of the most strict measures for the accuracy of a solution with respect to a reference solution is the energy norm of the error. The energy of the electromagnetic field corresponding to the error on the solution can be computed by integrating the square of the electric and magnetic fields everywhere outside the object. It is well-known that this energy is uniformly equivalent to the $H^{-\frac{1}{2}}(div;\Gamma)$ norm. The latter, however, only is meaningful for functions that are sufficiently smooth. In particular, functions with non-continuous normal components have infinite norm. The $H^{-\frac{1}{2}}(div;\Gamma)$ norms can thus not be used as a meaningful error criterion. In this contribution, a technique is introduced that allows the estimation of the near field error that is meaningful, even for solutions of the mortar element EFIE.

First the mortar element EFIE is revisited. Next, a technique to estimate the error in near field energy is elucidated. Finally, this technique is applied to the study of the accuracy of the mortar element EFIE.

2. The Mortar Element Electric Field Integral Equation

All fields and surfaces are assumed to have time dependency $e^{i\omega t}$. Consider a perfectly conducting scatterer with surface Γ with exterior normal $\boldsymbol{n}$. The scatterer is embedded in a background medium characterized by permittivity ϵ and permeability μ, corresponding to the wave number $k = \sqrt{\epsilon\mu}\omega$ and the impedance $\eta = \sqrt{\mu/\epsilon}$. Even though the theory explained here is valid in general, for clarity reasons it is assumed that Γ is the disjoint union of Γ_1 and Γ_2, which meet at the connected curve γ. The surfaces Γ_i are endowed with a triangular mesh $\mathcal{T}_i$ whit largest edge size h. The meshes on γ, inherited from $\mathcal{T}_i$ are denoted τ_i. Without loss of generality, assume that the finest of these meshes is τ_1. Denote the unit vector tangential to Γ_i, normal to γ, and exterior pointing w.r.t. Γ_i as $\boldsymbol{m}_i$. The structure is illuminated by an incident electric field $\boldsymbol{e}^i(\boldsymbol{r})$, which induced a current $\boldsymbol{j}(r)$ such that the resulting scattered field cancels the incident field at the boundary:

$$T(\boldsymbol{j})(\boldsymbol{r}) = \frac{1}{ik}\boldsymbol{n} \times \mathrm{grad}_\Gamma \int_\Gamma \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{div}_\Gamma \boldsymbol{j}(\boldsymbol{r}')\, d\boldsymbol{r}' - ik\boldsymbol{n} \times \int_\Gamma \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{j}(\boldsymbol{r}') d\boldsymbol{r}' = -\frac{1}{\eta}\boldsymbol{n} \times \boldsymbol{e}^i(\boldsymbol{r}) \tag{1}$$

In the mortar element EFIE, the induced current is sought after in the space $R_1 \times R_2$, where R_i is the space of RWG functions on Γ_i that have vanishing normal components on $\Gamma_i \setminus \gamma$. Let $\boldsymbol{f}_n$ denote the standard basis of this space. Note that this basis includes a number of *half RWG functions* corresponding to edges on γ. Equation (1) is tested by test functions $\boldsymbol{k}$ that are in the space spanned by $\boldsymbol{n} \times \boldsymbol{f}_n$. After partial integration, this yields

$$\begin{gathered} \frac{1}{ik}\int_{\Gamma\times\Gamma} \mathrm{div}_\Gamma \boldsymbol{k}(\boldsymbol{r}) \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{div}_\Gamma \boldsymbol{j}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r} - ik\int_{\Gamma\times\Gamma} \boldsymbol{k}(\boldsymbol{r}) \cdot \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{j}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r} \\ \sum_{i=1,2} \int_\gamma (\boldsymbol{m}_i \cdot \boldsymbol{k}_i(\boldsymbol{r}))\phi(\boldsymbol{r}) d\boldsymbol{r} = -\frac{1}{\eta}\int_{\Gamma\times\Gamma} \boldsymbol{k}(\boldsymbol{r}) \cdot \boldsymbol{e}^i(\boldsymbol{r}) d\boldsymbol{r}, \end{gathered} \tag{2}$$

where the Lagrange multiplier

$$\phi(\boldsymbol{r}) = \frac{1}{ik}\int_\Gamma \frac{e^{-ik|\boldsymbol{r}-\boldsymbol{r}'|}}{|\boldsymbol{r}-\boldsymbol{r}'|} \mathrm{div}_\Gamma \boldsymbol{j}(\boldsymbol{r}')\, d\boldsymbol{r}' \tag{3}$$

was introduced. The Lagrange multiplier will be sought for among the piecewise linear, globally continuous functions subordinate to the dual mesh to τ_1 [REF]. Finally, the weak discrete continuity of the candidate solution is imposed by testing the normal jump with testing functions ψ in the same space:

$$\sum_{i=1,2} \int_\gamma (\boldsymbol{m}_i \cdot \boldsymbol{j}_i(r))\psi(\boldsymbol{r})\, d\boldsymbol{r}. \tag{4}$$

3. Error Estimate

It a reference solution $\boldsymbol{j}_0(\boldsymbol{r})$ for the current density is available, a good measure for the error [3] is given by

$$\text{error}^2(\boldsymbol{j}) = \frac{1}{k}\int_{\Gamma\times\Gamma} \text{div}_\Gamma \boldsymbol{e}(\boldsymbol{r}) \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \text{div}_\Gamma \boldsymbol{e}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r} + k\int_{\Gamma\times\Gamma} \boldsymbol{e}(\boldsymbol{r}) \cdot \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{e}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r}, \tag{5}$$

where $\boldsymbol{e} = \boldsymbol{j}_0 - \boldsymbol{j}$. This error estimate is a good measure for the error in the near field energy and is much stricter than pointwise or root mean square errors on e.g. the far field pattern. Moreover, it respects the different scaling and importance of the solenoidal and non-solenoidal components of the current density. Given the expansion coefficients $\mathbf{x}$ and $\mathbf{x}_0$ of $\boldsymbol{j}$ and $\boldsymbol{j}_0$ with respect to a basis of RWG functions, (5) can be approximated by

$$\text{error}^2(\boldsymbol{j}) \approx (\mathbf{x}_0 - \mathbf{x})\mathbf{T}_\mathbf{0}(\mathbf{x}_\mathbf{0} - \mathbf{x}), \tag{6}$$

With $\mathbf{T}_{mn}$

$$\text{T}_{mn} = \frac{1}{k}\int_{\Gamma\times\Gamma} \text{div}_\Gamma \boldsymbol{f}_m(\boldsymbol{r}) \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \text{div}_\Gamma \boldsymbol{f}_n(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r} + k\int_{\Gamma\times\Gamma} \boldsymbol{f}_m(\boldsymbol{r}) \cdot \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{f}_n(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r}. \tag{7}$$

This approximation, however, is only valid for divergence conforming basis functions. It is not meaningful to apply (5) to the solution of the mortar element EFIE. In order to in a meaningful way estimate the near field error on the solution of the mortar element EFIE, the following procedure is being followed:

- Chose a closed smooth surface S that completely contains Γ. Define a triangular mesh $\mathcal{T}_S$ that is fine enough to allow for a high precision representation of the smooth scattered field.
- Compute the scattered field radiated by $\boldsymbol{j}$ at S and test it with the Buffa-Christiansen functions $\boldsymbol{g}_m^S$ [4] defined subordinate to the barycentric refinement of $\mathcal{T}_S$. Denote the array containing these testing coefficients $\mathbf{y}$.
- Compute and approximate expansion of the scattered field in terms of RWG functions $\boldsymbol{f}_m^S$ on $\mathcal{T}_S$ by projection, i.e. compute the array of RWG expansion coefficients $\mathbf{z} = \mathbf{G}^{-1}\mathbf{y}$, with $\mathbf{G}_{mn} = (\boldsymbol{n} \times \boldsymbol{g}_m^S, \boldsymbol{f}_m^S)$. Repeat all previous steps for the reference solution $\boldsymbol{j}_0$, giving rise to $\mathbf{z}_0$. The density $\boldsymbol{e}$ on S is the linear combination of RWG functions on $\mathcal{T}_S$ with expansion coefficients $\boldsymbol{z}_0 - \boldsymbol{z}$.
- Compute the estimate:

$$\text{error}^2(\boldsymbol{j}) = \frac{1}{k}\int_{\text{S}\times\text{S}} \text{div}_\text{S} \boldsymbol{e}(\boldsymbol{r}) \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \text{div}_\text{S} \boldsymbol{e}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r} + k\int_{\text{S}\times\text{S}} \boldsymbol{e}(\boldsymbol{r}) \cdot \frac{1}{|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{e}(\boldsymbol{r}')\, d\boldsymbol{r}' d\boldsymbol{r}, \tag{8}$$

Since S is smooth, $\mathcal{T}_S$ conforming, and the mesh size is chosen small, all steps in this procedure are much more accurate than the solution of the mortar element EFIE itself. Moreover, this process makes sense for conforming and non-conforming current densities alike.

Numerical Results

Consider the geometry in Fig. 1: a square 1m by 1m plate that is subdivided in two halves. A series of non-conforming meshes is prepared with the mesh parameters from table I.

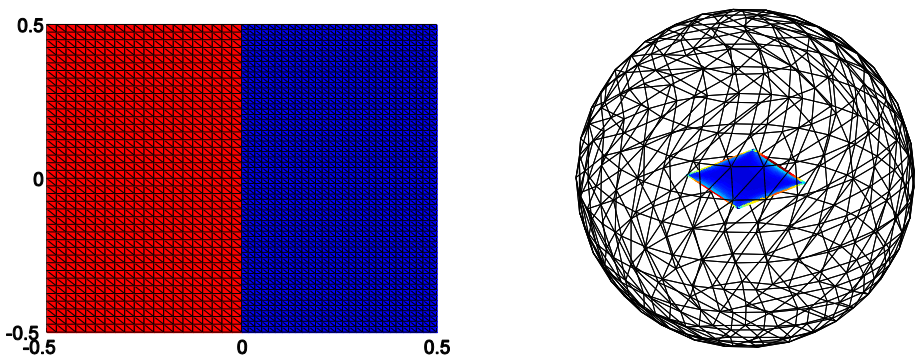

Figure 1: Left: Scattering surface: a 1m by 1m square planer metallic sheet dived in two subcomponents. The subcomponents are endowed with meshes that are not compatible at their common boundary. **Right:** Location of the near field surface where the estimation of the error will take place, in this case a sphere of radius 2m. Note that the triangulation of the sphere can be much coarser because both the sphere and the scattered field at the sphere are smooth.

Table 1: Mesh sizes in meter used to discretize the structure in Fig. 1

Left	0.25	0.20	0.15	0.10	0.05	0.025
Right	0.175	0.14	0.105	0.070	0.035	0.0175

The scatterer is illuminated by a plane wave of amplitude 1V/m, wavelength 6.28m, direction of travel (0.7007, 0.0982, 0.7066) m, and polarization (-0.1388, 0.9903, 0) m. For each combination of mesh parameters, the mortar element EFIE is solved, and the error compared to a reference solution obtained by solving the classical EFIE at a uniform mesh parameter of 0.025m is computed by applying the procedure introduced above. The resulting relative errors can be read from Fig. 2.

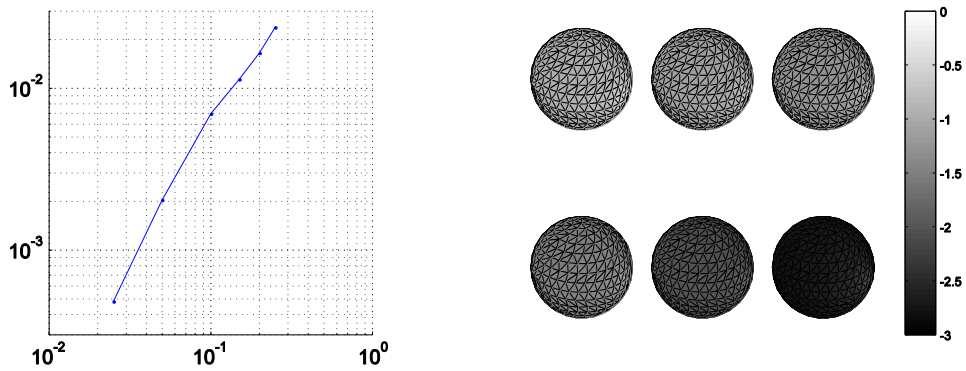

Figure 2: **Left:** The 10-logarithm relative error vs the 10-logarithm of the mesh parameter of the left half of the scatterer. The convergence of the near field is approximately quadratic. **Right:** Surface plots of the error on the scattered field for the different mesh parameters.

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