Wireless Network Resource Allocation Optimization by Nonlinear Perron-Frobenius Theory

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Abstract

A basic question in wireless networking is how to optimize the wireless network resource allocation for utility maximization and interference management. In this paper, we present an overview of a Perron-Frobenius theoretic framework to overcome the notorious non-convexity barriers in wireless utility maximization problems. Through this approach, the optimal value and solution of the optimization problems can be analytically characterized by the spectral property of matrices induced by nonlinear positive mappings. It also provides a systematic way to derive distributed and fast-convergent algorithms and to evaluate the fairness of resource allocation. This approach can even solve several previously open problems in the wireless networking literature, e.g., Kandukuri and Boyd (TWC 2002), Wiesel, Eldar and Shamai (TSP 2006), Krishnan and Luss (WCNC 2011). More generally, this approach links fundamental results in nonnegative matrix theory and (linear and nonlinear) Perron-Frobenius theory with the solvability of non-convex problems.

I. Introduction

A remarkable feature of the growth in wireless data usage is that many new devices today are operating in the wireless spectrum that are meant to be shared among many different users. Yet, the sharing of the spectrum is far from perfect. Due to the broadcast nature of the wireless medium, interference is a major source of performance impairment. Current systems suffer from deteriorating quality due to fixed resource allocation that fails to consider interference. As wireless networks are becoming more heterogeneous and ubiquitous in our life, they are also becoming more difficult to design and manage. How should these large complex wireless net-works be analyzed and designed with clearly-defined fairness and optimality in mind? Wireless network optimization has become an important tool to design resource allocation algorithms to realize the untapped benefits of wireless resources and to manage interference in wireless networks [1], [2]. Without appropriate resource coordination, the network can become unstable or may operate in a highly inefficient and unfair manner.

Important objectives in wireless networks can be modeled by nonlinear utility functions of wireless link metrics, e.g., signal-to-interference-and-noise ratio (SINR), meansquare error (MSE) and the outage probability. The total network utility is then maximized over the joint solution space of all possible wireless link metrics, e.g., powers and interference. This can be used to address issues such as understanding how wireless network algorithms can interact between network layers, e.g., physical and medium access control layers, to achieve provable efficiency for the overall system or understanding how fairness permeates through network layers when interference is dominant. This can open up new opportunities to jointly optimize physical layer innovation and other networking control mechanism, and can lead to design wireless network protocols.

The main challenges of solving these wireless utility maximization problems come from the nonlinear dependency of link metrics on channel conditions and powers, as well as possible interference among the users. These are nonconvex problems that are notoriously difficult to solve, and designing scalable algorithms with low complexity to solve them optimally is even harder. In this paper, we will present a new theoretical foundation for solving network optimization problems in wireless networks. The goal is to present a suite of theory and algorithms based on the nonlinear Perron-Frobenius theory[1] to optimize performance metrics in wireless networks and to design efficient algorithms with low complexity that are applicable to a wide range of wireless network applications. This approach also resolves open issues related to algorithm design in [8] for a worst outage probability problem, the max-min SINR problems for beamforming in [9] and for small cells in [10]. This paper is organized as follows. In Section II, we introduce the wireless network model. In Section III, we present how the Perron-Frobenius theory is used for solving wireless max-min fairness optimization problems. We conclude the paper in Section IV. The following notation is used. Boldface uppercase letters denote matrices, boldface lowercase letters denote column vectors and italics denote scalars.

II. System Model

Consider a multiuser communication system with L users (logical transmitter/receiver pairs) sharing a common frequency. Each user employs a singe-user decoder, i.e., treating interference as additive Gaussian noise, and has perfect channel state information at the receiver. Assume a noise power n_l at the lth receiver. At each transmitter, the signal is constrained by

[1]In nonnegative matrix theory, the classical linear Perron-Frobenius theorem is an important result that concerns the eigenvalue problem of nonnegative matrices, and has many engineering applications [3], [4], [5]. Efforts to extend the linear Perron-Frobenius theorem to nonlinear ones (to study the dynamics of cone-preserving operators) have been intensive, and the nonlinear Perron-Frobenius theory is now emerging to provide a mathematically rigorous and practically relevant technique to engineering problems [6], [7].

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an average power constraint p_l. The vector $(p_1, \dots, p_L)^\top$ is the transmit power vector, which is the optimization variable of interest in this paper. The power received from the jth transmitter at lth receiver is given by $G_{lj}p_j$ where G_{lj} represents the nonnegative path gain between the jth transmitter and the lth receiver (it may also encompass antenna gain and coding gain) that is often modeled as proportional to $d_{lj}^{-\gamma}$, where d_{lj} denotes distance, γ is the power fall-off factor.

There are several important performance metrics in wireless networks. The signal-to-interference-and-noise ratio (SINR) is an example that measures the quality of service for link transmission. Specifically, in a static channel (or relatively slow channel fading), this metric for the lth user can be given by [11]:

$$\mathsf{SINR}_l(\mathbf{p}) = \frac{G_{ll}p_l}{\sum_{j \neq l} G_{lj}p_j + n_l}. \tag{1}$$

There are many types of nonlinear power constraints that are imposed either due to regulatory policy for health consideration or to limit excessive levels of interference to macrocell users. For example, a wireless cognitive radio network has interference temperature to limit interference from secondary users. In heterogeneous wireless networks, small cell network coverage is limited so that macro-cell users' performance are guaranteed [10].

III. Wireless Max-min Fairness Optimization

In this section, we introduce a general framework for max-min fairness optimization in wireless networks that can be applied to a variety of wireless network applications. The max-min fairness is an egalitarian fairness guarantee to protect the worst case performance in the network.

A. Problem statement

Let us consider a class of utility functions that satisfy the following assumptions.

Assumption 1 (Competitive Utility Functions):

- *Positivity*: For all i, $u_i(\mathbf{p}) > 0$ if $\mathbf{p} > \mathbf{0}$ and, in addition, $u_i(\mathbf{p}) = 0$ if and only if $p_i = 0$.
- *Competitiveness*: For all i, u_i is strictly increasing with respect to p_i and is strictly decreasing with respect to p_j, for $j \neq i$, when $p_i > 0$.
- *Directional Monotonicity*: For $\lambda > 1$ and $p_i > 0$, $u_i(\lambda \mathbf{p}) > u_i(\mathbf{p})$, for all i.

For the utility functions, the competitiveness assumption models the interaction between users in a wireless network and the directional monotonicity captures the increase in utility as the total power consumption increases.

To optimize max-min utility fairness to all the users, the wireless network optimization can be formulated as

$$\begin{array}{ll} \text{maximize} & \min\limits_{i=1,\dots,L} \; u_i(\mathbf{p}) \\ \text{subject to} & \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}} \\ \text{variables:} & \mathbf{p}, \end{array} \tag{2}$$

where $\mathbf{g}(\mathbf{p})$ is a general performance constraint set, and $\bar{\mathbf{g}} = [\bar{g_1}, \dots, \bar{g_K}]^T$ is the vector of constraint values. Let the optimal solution to (2) be $\mathbf{p}^*$.

In (2), $u_l(\mathbf{p})$ can be a general performance metric in terms of SINR of the lth link. For example, let $u_l(\mathbf{p}) = \gamma_l / \mathsf{SINR}_l(\mathbf{p})$ (see Section III-D below), which is then the max-min weighted SINR problem [9], [12], [13]. Another example is when the SINR is a random variable and to let $u_l(\mathbf{p}) = 1/\text{Prob}(\mathsf{SINR}_l(\mathbf{p}) < \gamma_l)$, which is the inverse of the outage probability for the event that the SINR of the lth link falls below a given threshold γ_l, and this is the worst outage probability problem [8], [14].

By introducing an auxiliary variable τ, solving (2) is equivalent to solving the following optimization problem:

$$\begin{array}{ll} \text{maximize} & \tau \\ \text{subject to} & u_i(\mathbf{p}) \geq \tau, \quad \text{for } i = 1, \dots, L, \\ & g_k(\mathbf{p}) \leq \bar{g}_k, \quad \text{for } k = 1, \dots, K, \\ \text{variables:} & \mathbf{p}, \; \tau. \end{array} \tag{3}$$

B. Solution Methodology

Solving (2) is generally difficult and, in the following, we present how to solve (2) with specifically the vector of constraint functions $\mathbf{g}(\mathbf{p}) = [g_1(\mathbf{p}), \dots, g_K(\mathbf{p})]^T$ and $\bar{\mathbf{g}}$ that describe the set of monotonic constraints satisfying the following assumptions (see [15] for more details).

Assumption 2 (Monotonic Constraints):

- *Strict Monotonicity*: For all k, $g_k(\mathbf{p}_1) > g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 > \mathbf{p}_2$, and $g_k(\mathbf{p}_1) \geqq g_k(\mathbf{p}_2)$ if $\mathbf{p}_1 \geqq \mathbf{p}_2$.
- *Feasibility*: The set $\{\mathbf{p} > \mathbf{0} : \mathbf{g}(\mathbf{p}) \leqq \bar{\mathbf{g}}\}$ is non-empty.
- *Validity*: For any $\mathbf{p} > \mathbf{0}$, there exists $\lambda > 0$ such that $g_k(\lambda \mathbf{p}) \geqq \bar{g}_k$, for some k.

The strict monotonicity captures the increase in cost or resource consumption as $\mathbf{p}$ increases, the feasibility ensures that there exists a positive power vector in the feasible set, and the validity ensures that the set of constraints is meaningful. If the validity condition does not hold, the corresponding constraint can be simply removed without loss of generality.

Lemma 1: For $\{u_i\}_{i=1}^L$, $\{g_k\}_{k=1}^K$, and $\{\bar{g}_k\}_{k=1}^K$ that satisfy Assumptions 1 and 2, the optimal solution $(\tau^*, \mathbf{p}^*)$ in (3) is positive, i.e., $\tau^* > 0$ and $\mathbf{p}^* > \mathbf{0}$, and, at optimality, all the L constraints of $u_i(\mathbf{p}) \geq \tau$ are tight and at least one of the K constraints of $g_k(\mathbf{p}) \leq \bar{g}_k$ is active. That is, $u_i(\mathbf{p}^*) = \tau^*$, for all i and $g_k(\mathbf{p}^*) = \bar{g}_k$, for some k (let us denote this constraint set for $\mathbf{p}$ by $\mathcal{U}$).

By Lemma 1, it follows that

$$\frac{1}{\tau^*} p_i^* = \frac{1}{u_i(\mathbf{p}^*)} p_i^* \triangleq T_i(\mathbf{p}^*). \tag{4}$$

This means that $\mathbf{p}^*$ is a solution to the fixed point equation

$$\frac{1}{\tau^*}\mathbf{p}^* = [T_1(\mathbf{p}^*), \ldots, T_L(\mathbf{p}^*)]^T \triangleq \mathbf{T}(\mathbf{p}^*). \tag{5}$$

Definition 1: The function $\beta : \mathbb{R}_+^L \to \mathbb{R}_+$ of $\mathbf{p}$ (called the *scale* of $\mathbf{p}$) is defined as

$$\beta(\mathbf{p}) \quad \triangleq \min\{\beta' \geqq 0 : g_k(\mathbf{p}/\beta') \leqq \bar{g}_k, \forall k\}. \tag{6}$$

Lemma 2: The scale $\beta : \mathbb{R}_+^L \to \mathbb{R}_+$ defined in Definition 1 satisfies the following properties:

1) β is not identically zero and, in fact, $\beta(\mathbf{p}) > 0$, for all $\mathbf{p} > \mathbf{0}$;
2) $\beta(\lambda\mathbf{p}) = \lambda\beta(\mathbf{p})$ for $\mathbf{p} \geqq \mathbf{0}$ and $\lambda \geqq 0$ (i.e., positively homogeneous);
3) $\mathbf{0} \leqq \mathbf{p} \leqq \mathbf{q}$ implies $\beta(\mathbf{p}) \leqq \beta(\mathbf{q})$ (i.e., monotonic).

It follows that the solution of the optimization problem in (3) (and, thus, (2)) is a solution of a conditional eigenvalue problem where the objective is to find $(\tau^*, \mathbf{p}^*)$ such that [15]:

$$\frac{1}{\tau^*}\mathbf{p}^* = \mathbf{T}(\mathbf{p}^*), \;\; \tau^* \in \mathcal{R}, \;\; \mathbf{p}^* \in \mathcal{U}. \tag{7}$$

C. Algorithm

Here, we use $\mathbf{p}(t)$ to represent the power vector obtained in the t-th iteration of the algorithm.

Algorithm 1 (Monotonic Constrained Max-Min Utility):

1) Initialize power vector $\mathbf{p}(0) > \mathbf{0}$.
2) Update power vector $\mathbf{p}(t+1)$:

$$p_i(t+1) = \frac{p_i(t)}{u_i(\mathbf{p}(t))} \left(\triangleq T_i(\mathbf{p}(t))\right), \;\; \forall i.$$

3) Scale power vector $\mathbf{p}(t+1)$:

$$\mathbf{p}(t+1) \leftarrow \frac{\mathbf{p}(t+1)}{\beta(\mathbf{p}(t+1))}.$$

4) Repeat Steps 2 and 3 until convergence.

The following theorem establishes the existence and the uniqueness of the solution of the conditional eigenvalue problem in (7) as well as the convergence of Algorithm 1 [15].

Theorem 1: Suppose that $\mathbf{T} : \mathcal{R}_+^L \to \mathcal{R}_+^L$, defined as $\mathbf{T}(\mathbf{p}) = [T_1(\mathbf{p}), \ldots, T_L(\mathbf{p})]^\top$, where $T_i(\mathbf{p}) = p_i/u_i(\mathbf{p})$, satisfies the following conditions: (i) there exist numbers $a > 0$, $b > 0$, and a vector $\mathbf{e} > \mathbf{0}$ such that $a\mathbf{e} \leqq \mathbf{T}(\mathbf{p}) \leqq b\mathbf{e}$, for all $\mathbf{p} \in \mathcal{U}$; (ii) for any $\mathbf{p}$, $\mathbf{q} \in \mathcal{U}$ and $0 \leqq \lambda \leqq 1$: If $\lambda\mathbf{p} \leqq \mathbf{q}$, then $\lambda\mathbf{T}(\mathbf{p}) \leqq \mathbf{T}(\mathbf{q})$; and, if $\lambda\mathbf{p} \leq \mathbf{q}$ with $\lambda < 1$, then $\lambda\mathbf{T}(\mathbf{p}) < \mathbf{T}(\mathbf{q})$. Then, the following properties hold:

(a) The conditional eigenvalue problem in (7) has a unique solution $\mathbf{p}^* \in \mathcal{U}$ and $\tau^* > 0$.
(b) The power vector $\mathbf{p}(t)$ in Algorithm 1 converges to $\mathbf{p}^*$ (i.e., the solution of (7) and, thus, (2)) for any initial point $\mathbf{p}(0) \geqq \mathbf{0}$ with $\beta(\mathbf{T}(\mathbf{p}(0))) > 0$.

Starting from any initial point $\mathbf{p}(0)$, $\mathbf{p}(t)$ in Algorithm 1 converges geometrically fast. A notable special case of Theorem 1 is when $\mathbf{T}(\mathbf{p}) = \mathbf{F}\mathbf{p}$ where $\mathbf{F}$ is an irreducible[2] nonnegative square matrix, and this is the classical linear Perron-Frobenius theory in linear algebra [5].[3] Then, $1/\tau^* = \rho(\mathbf{F})$ and $\mathbf{p}^*$ is the right eigenvector of $\rho(\mathbf{F})$ in (7) for this special case.

[2]A nonnegative matrix $\mathbf{F}$ is said to be irreducible if there exists a positive integer m such that the matrix $\mathbf{F}^m$ has all entries positive.

[3]The Perron-Frobenius eigenvalue of an irreducible nonnegative matrix $\mathbf{F}$ is the spectral radius (eigenvalue with the largest absolute value) of $\mathbf{F}$ denoted by $\rho(\mathbf{F})$, and the Perron (right) and left eigenvector of $\mathbf{F}$ associated with $\rho(\mathbf{F})$ are denoted by $\mathbf{x}(\mathbf{F}) \geq \mathbf{0}$ and $\mathbf{y}(\mathbf{F}) \geq \mathbf{0}$ respectively. Furthermore, $\rho(\mathbf{F})$ is simple and positive, and $\mathbf{x}(\mathbf{F}), \mathbf{y}(\mathbf{F}) > \mathbf{0}$ (cf. [3], [4], [5]).

Another notable special case of Theorem 1 is when $\mathbf{T}(\mathbf{p})$ is *concave* [6] (the *hidden convexity* to be exploited in solving a seemingly nonconvex (2)). As a by-product of applying Theorem 1, we have in [13] resolved an open issue in algorithm design in [9], and we have in [14] resolved an open issue in algorithm design for the interference-limited case in [8].

D. Example: Max-min Weighted SINR

We illustrate an example of using Theorem 1 and Algorithm 1 to maximize the worst case SINR given as follows:

$$\begin{array}{ll} \text{maximize} & \min\limits_{l} \dfrac{\mathsf{SINR}_l(\mathbf{p})}{\gamma_l} \\ \text{subject to} & \mathbf{a}^T\mathbf{p} \le 1,\ \mathbf{p} \ge \mathbf{0} \\ \text{variables:} & \mathbf{p}, \end{array} \tag{8}$$

where $\mathsf{SINR}_l(\mathbf{p})$ is given in (1), and $\mathbf{a}$ is some given positive vector (examples in cellular and heterogeneous femtocell networks can be found in [9], [16], [17] and [10]). There are a few approaches to solving (8). For example, (8) can be reformulated as a geometric program, and be solved numerically using the interior point method [2], [18]. We show how Theorem 1 can solve (8). Since $\mathbf{a}^T\mathbf{p} \le 1$ is tight at optimality, solving (8) is equivalent to solving for the unknown $(\tau, \mathbf{p})$ in the fixed-point problem: $\frac{1}{\tau}\mathbf{p} = \mathbf{F}\mathbf{p} + \mathbf{v},\ \ \mathbf{a}^T\mathbf{p} = 1$ where the nonnegative matrix $\mathbf{F}$ has entries $F_{lj} = \gamma_l G_{lj}/G_{ll}$ for $l \neq j$ and 0 otherwise, and $v_l = \gamma_l n_l / G_{ll}$.

Now, by invoking Theorem 1 for $\mathbf{T}(\mathbf{p}) = \mathbf{F}\mathbf{p} + \mathbf{v}$ (notice that this is affine and hence concave), we deduce that the optimal value and solution to (8) is $1/\rho(\mathbf{F} + \mathbf{v}\mathbf{a}^T)$ and the right eigenvector of $\mathbf{F} + \mathbf{v}\mathbf{a}^T$ respectively. Furthermore, this resolves an open issue of algorithm design in [10].

IV. Conclusion

We have presented an advanced suite of theory and algorithms based on the nonlinear Perron-Frobenius theory to solve a class of max-min fairness optimization problems and nonconvex utility maximization problems that find applications for resource allocation and interference management in a wide variety of wireless networks such as cellular networks and cognitive radio networks. The nonlinear Perron-Frobenius theory characterizes the optimal solution analytically, and provides a systematic way to derive distributed fast algorithms that cover power control, rate control, antenna beamforming and cross-layer design.

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