HIGH-FREQUENCY SCATTERING BY ELONGATED BODIES

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1 Introduction

The specifics of the diffraction process by elongated objects is known to be in the low attenuation of creeping waves running along the surface. This also affects the far field. The usual approach to the problems of scattering by spheroids is based on the field decomposition by spheroidal functions (see e.g. [1]). However this approach faces some difficulties in the case of high frequencies and large eccentricities. Only recently [2] the limitation of $b/a \leq 10$ was overcome. Moreover the case of axial or near axial incidence is not considered, probably, because of numerical instabilities.

We use another approach based on the results of [3, 4] where the high-frequency asymptotics of the currents induced on the surface is derived, in [3] for axial incidence and in [4] for the incidence at a small angle to the axis. In both cases the body is assumed to be circular symmetric and well approximated by the spheroid with appropriately chosen minor a and major b semiaxes. The asymptotic decomposition in inverse powers of large parameter kb, where k is the wave number, depends on the parameter $\chi = ka^2/b$ which characterizes the rate of elongation and is assume to be of the order of unity. The incidence angle ϑ is presented in this asymptotics via another parameter $\beta = \sqrt{kb}\,\vartheta$. For axial incidence the leading order term of the asymptotics gives a very good approximation for the induced current as was checked in [5] by comparison with numerical test results. The induced currents play the role of passive sources for the secondary field and allow the far field asymptotics to be studied. For axial case the total effective scattering cross-section was derived in [6]. Here we present formulae for near axial incidence.

2 Induced currents

Consider a plane wave with the wavenumber k incident on an elongated spheroid with semiaxes a and b at a small angle ϑ to its axis. Our goal is to find the asymptotics of the diffracted field under the assumption that kb is large, while

$$\chi \equiv \frac{ka^2}{b} = O(1), \qquad \text{and} \qquad \beta \equiv \sqrt{kb}\vartheta = O(1). \tag{1}$$

In the boundary layer near the surface we use coordinates (η, τ, φ) introduced by the formulae

$$r = a\sqrt{1-\eta^2}\sqrt{\tau}, \qquad z = b\eta + \frac{a^2}{2b}(\tau - 1)\eta \tag{2}$$

relating them to cylindrical coordinates (r, φ, z). Actually, η is the usual angular spheroidal coordinate and τ is the stretched radial coordinate, such that $\tau = 1$ on the surface and $\tau = 0$ on the axis.

The incident wave can be represented as a sum of TE and TM waves. Due to the linearity of the problem we can consider the cases of TE and TM polarizations separately. However the formulae are similar and to distinguish these two cases we use superscripts $+1$ for TE case and -1 for TM case. We express all the components of the electric and magnetic vectors via E_φ and H_φ components. The latter we represent in the form of Fourier series by the angle φ. According to the symmetry of the problem we have

$$E_\varphi^{+1} = \sum_{n=0}^{\infty} i^{n+1}\Big(P_{n+1}^{+1,+1} - P_{n-1}^{+1,-1}\Big)\cos(n\varphi), \qquad H_\varphi^{+1} = \sum_{n=1}^{\infty} i^{n-1}\Big(P_{n+1}^{+1,+1} + P_{n-1}^{+1,-1}\Big)\sin(n\varphi). \tag{3}$$

$$E_\varphi^{-1} = \sum_{n=1}^{\infty} i^{n+1}\Big(P_{n+1}^{-1,+1} + P_{n-1}^{-1,-1}\Big)\sin(n\varphi), \qquad H_\varphi^{-1} = \sum_{n=0}^{\infty} i^{n-1}\Big(-P_{n+1}^{-1,+1} + P_{n-1}^{-1,-1}\Big)\cos(n\varphi). \tag{4}$$

Further, following the parabolic equation method, we extract oscillating factor in the form of $\exp(ikb\eta)$ and get the system of parabolic equations. Due to the use of $P_n^{\pm1,\pm1}$ this system is uncoupled. It can be solved by variables separation method which results in the following representations

$$P_n^{s_1,s_2}(\eta,\tau) = \frac{e^{ikb\eta - ik\chi\eta/2}}{\pi\sqrt{\chi\eta}\sqrt{1-\eta^2}} \int_{-\infty}^{+\infty} \left(\frac{1-\eta}{1+\eta}\right)^{i\lambda} c_n(\lambda,\beta)\times$$
$$\times \Big(M_{i\lambda,n/2}(-i\chi\tau) + r_n^{s_1,s_2}(\lambda,\beta)W_{i\lambda,n/2}(-i\chi\tau)\Big)d\lambda, \quad n = 0, 1, \ldots \tag{5}$$

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involving Whittaker functions M and W. Here and below we assume the functions P with negative subscripts are identically equal to zero and introduce indices s_1 and s_2 which take the values of ± 1.

The terms with Whittaker function M in (5) represent the incident field matching to which results in an integral equation for the amplitudes c_n. It can be solved by means of Mellin transform which defines

$$c_n(\lambda,\beta) = \frac{\Gamma\left(\frac{n}{2}+\frac{1}{2}+i\lambda\right)\Gamma\left(\frac{n}{2}+\frac{1}{2}-i\lambda\right)}{\Gamma^2(n+1)}\frac{M_{i\lambda,n/2}(i\beta^2)}{\beta}. \tag{6}$$

The terms with Whittaker function W in (5) correspond to the secondary field, and the coefficients $r_n^{s_1,s_2}$ play the role of reflection coefficients and can be defined from the boundary conditions:

$$r_1^{+1,+1} = -\frac{M_{i\lambda,1/2}(-i\chi)}{W_{i\lambda,1/2}(-i\chi)}, \qquad r_1^{-1,+1} = -\frac{\dot{M}_{i\lambda,1/2}(-i\chi)}{\dot{W}_{i\lambda,1/2}(-i\chi)}, \tag{7}$$

$$r_n^{s_1,s_2} = -\frac{1}{Z_{n-s_2}}\Big\{M_{i\lambda,\frac{n}{2}}(-i\chi)\dot{W}_{i\lambda,\frac{n}{2}-s_2}(-i\chi) + \dot{M}_{i\lambda,\frac{n}{2}}(-i\chi)W_{i\lambda,\frac{n}{2}-s_2}(-i\chi) + s_1\frac{\Gamma(n+1-2s_2)}{\Gamma\left(\frac{n+1}{2}-s_2-i\lambda\right)}\frac{c_{n-2s_2}(\lambda,\beta)}{c_n(\lambda,\beta)}\Big\}, \qquad n \geq 1+s_2, \tag{8}$$

where dot over a function denotes its derivative and

$$Z_n = W_{i\lambda,\frac{n-1}{2}}(-i\chi)\dot{W}_{i\lambda,\frac{n+1}{2}}(-i\chi) + \dot{W}_{i\lambda,\frac{n-1}{2}}(-i\chi)W_{i\lambda,\frac{n+1}{2}}(-i\chi). \tag{9}$$

The total field in the boundary layer near the surface is the sum of the primary wave running in the positive direction of z and waves that are formed when the primary wave encircles the end of the spheroid. The asymptotics (5) describes only the primary wave and is valid in the middle part of the spheroid. Reflected backward wave asymptotics is found in [7] for axial incidence. For $\vartheta \neq 0$ backward wave asymptotics can be found by the same approach. However it does not contribute to forward scattering.

3 The scattered field

Stratton-Chu formula [8] expresses the scattered field via the currents J on the surface. For the perfect conductor it reduces to

$$\mathbf{H}^s = -\frac{1}{4\pi}\iint \mathbf{J}\times\nabla G\, dS, \qquad G(\mathbf{r},\mathbf{r}_0) = \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{|\mathbf{r}-\mathbf{r}_0|}. \tag{10}$$

By moving the observation point $\mathbf{r}_0$ to infinity we get the formula for the far field amplitude

$$\mathbf{\Psi} = -\frac{1}{4\pi}\iint \mathbf{J}\times\nabla\psi\, dS, \tag{11}$$

where ψ is the far field amplitude of G. Due to the reciprocity ψ coincides with the plane wave incident on the spheroid from the opposite direction. In the way similar to the previous section it can be represented in the boundary layer near the surface as [9]

$$\psi = \frac{1}{2}\psi_0 + \sum_{m=1}^{+\infty}\psi_m\cos[m(\varphi-\varphi_0)], \tag{12}$$

$$\psi_m = \frac{2i^n e^{-ikb\eta+i\chi\eta/2}}{\pi\sqrt{1-\eta^2}\sqrt{\chi\tau}}\int_{-\infty}^{+\infty}\left(\frac{1+\eta}{1-\eta}\right)^{i\mu} c_m(\mu,\beta_0)M_{i\mu,\frac{m}{2}}(-i\chi\tau)d\mu, \tag{13}$$

where $\beta_0 = \sqrt{kb}\vartheta_0$ is the scaled observation angle.

Substituting the asymptotics of the field in the boundary layer presented by formulae (3)–(9) and the representation (12), (13) into (11) we get a cumbersome expression. However due to the orthogonality of trigonometric functions and the formula

$$\int_{-1}^{1}\left(\frac{1-\eta}{1+\eta}\right)^{i(\lambda-\mu)}\frac{d\eta}{1-\eta^2} = \pi\delta(\lambda-\mu) \tag{14}$$

it can be simplified and we get relatively simple expressions

$$\Psi_x^{+1} = \sum_{n=0}^{+\infty} \Psi_{nx}^{+1} \cos(n\varphi_0), \qquad \Psi_y^{+1} = \sum_{n=1}^{+\infty} \Psi_{ny}^{+1} \sin(n\varphi_0), \tag{15}$$

$$\Psi_x^{-1} = \sum_{n=1}^{+\infty} \Psi_{nx}^{-1} \sin(n\varphi_0), \qquad \Psi_y^{-1} = \sum_{n=0}^{+\infty} \Psi_{ny}^{-1} \cos(n\varphi_0), \tag{16}$$

where

$$\Psi_{nx}^{s_1} = \frac{ib}{\pi}(-1)^{n+1} \int\limits_{-\infty}^{+\infty} c_n(\lambda,\beta) c_n(\lambda,\beta_0) \Big(s_1 r_n^{s_1,-1}(\lambda) + r_n^{s_1,+1}(\lambda) \Big) \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-i\lambda\right)} d\lambda, \tag{17}$$

$$\Psi_{ny}^{s_1} = \frac{ib}{\pi}(-1)^{n+1} \int\limits_{-\infty}^{+\infty} c_n(\lambda,\beta) c_n(\lambda,\beta_0) \Big(s_1 r_n^{s_1,-1}(\lambda) - r_n^{s_1,+1}(\lambda) \Big) \frac{\Gamma(n+1)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}-i\lambda\right)} d\lambda. \tag{18}$$

The reflection coefficients $r_n^{s_1,s_2}$ are defined by formulae (7), (8) and it is accepted that $r_0^{s_1,+1} \equiv 0$.

4 The total scattering cross-section

Formulae (17), (18) give the asymptotics of the far field amplitude of the magnetic field under the assumption that the angles ϑ and ϑ_0 are small, that is the directions of the incidence and of the observation are in a narrow cones near the axis of the body. Nevertheless these formulae allow the total scattering cross-section $\sigma(\vartheta_0)$ to be found. According to the "optical" theorem

$$\sigma(\vartheta_0) = \frac{4\pi}{k} \mathrm{Im} \left\langle \mathbf{\Psi}(\vartheta_0, \vartheta_0, 0), \mathbf{h} \right\rangle, \tag{19}$$

where $\mathbf{h}$ is the magnetic polarization vector of the incident wave and angular brackets denote scalar product.

To exclude the dependence of the effective cross-section on the size of the spheroid we normalize it by the visible cross-section $\sigma_0 = \pi a^2 \sqrt{1+\beta_0^2/\chi}$. Then

$$\Sigma^{\pm} \equiv \frac{\sigma^{\pm}}{\sigma_0} = \frac{4}{\pi\chi\sqrt{1+\frac{\beta_0^2}{\chi}}} \sum_{n=0}^{+\infty} (-1)^n \mathrm{Re} \left(\int\limits_{-\infty}^{+\infty} c_n^2(\lambda,\beta_0) \Big(r_n^{\pm,+1} + r_n^{\pm,-1} \Big) \frac{\Gamma(n+1)}{\Gamma\left(\frac{n+1}{2}-i\lambda\right)} \, d\lambda \right). \tag{20}$$

Substituting here expressions for the reflection coefficients $r_n^{s_1,s_2}$ and expressing Whittaker functions via Coulomb wave functions F and H^+, we finally get

$$\Sigma^{\pm} = -\frac{32}{\pi\chi\beta^2\sqrt{1+\beta^2/\chi}} \mathrm{Im} \left(\int\limits_{-\infty}^{+\infty} \left\{ \frac{\delta_{\pm} F_0\left(-\lambda, \frac{\chi}{2}\right)}{\delta_{\pm} H_0^+\left(-\lambda, \frac{\chi}{2}\right)} F_0^2\left(\lambda, \frac{\beta^2}{2}\right) \right.\right.$$
$$\left.\left. + \sum_{n=0}^{+\infty} \frac{1}{P_n} \left[F_{\frac{n-1}{2}}^2\left(\lambda, \frac{\beta^2}{2}\right) Q_n + F_{\frac{n+1}{2}}^2\left(\lambda, \frac{\beta^2}{2}\right) Q_n \pm 2 F_{\frac{n-1}{2}}\left(\lambda, \frac{\beta^2}{2}\right) F_{\frac{n+1}{2}}\left(\lambda, \frac{\beta^2}{2}\right) \right] \right\} d\lambda \right), \tag{21}$$

where $\delta_+ = 1$, $\delta_- = \frac{d}{d\chi}$ and the following quantities are introduced

$$P_n = H_{\frac{n-1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right) \dot{H}_{\frac{n+1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right) + \dot{H}_{\frac{n-1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right) H_{\frac{n+1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right), \tag{22}$$

$$Q_n = F_{\frac{n-1}{2}}\left(-\lambda, \frac{\chi}{2}\right) \dot{H}_{\frac{n+1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right) + \dot{F}_{\frac{n-1}{2}}\left(-\lambda, \frac{\chi}{2}\right) H_{\frac{n+1}{2}}^+\left(-\lambda, \frac{\chi}{2}\right). \tag{23}$$

The results of computations according to the formula (21) are presented on the figure. It shows that the rate of elongation significantly influences the scattering characteristics for axial and near axial incidence. The classical high frequency limit is approached at very high wavesizes of the body (for $a : b = 1 : 20$ at $kb \sim 10^4$) which are hardly reachable even by modern numerical approaches.

We considered only the case of plane wave scattering, but the approach allows other types of incident fields, for example point sources or Gaussian beams, to be studied. In the asymptotic formulae this will only modify the amplitudes $c_n(\lambda)$. Contrarily, generalizations from the case of the perfectly conducting surface to the impedance or dielectric bodies are more difficult and require next order terms of the asymptotics to be found. These next order terms also introduce corrections due to the deviation of the shape of the body from the spheroidal.

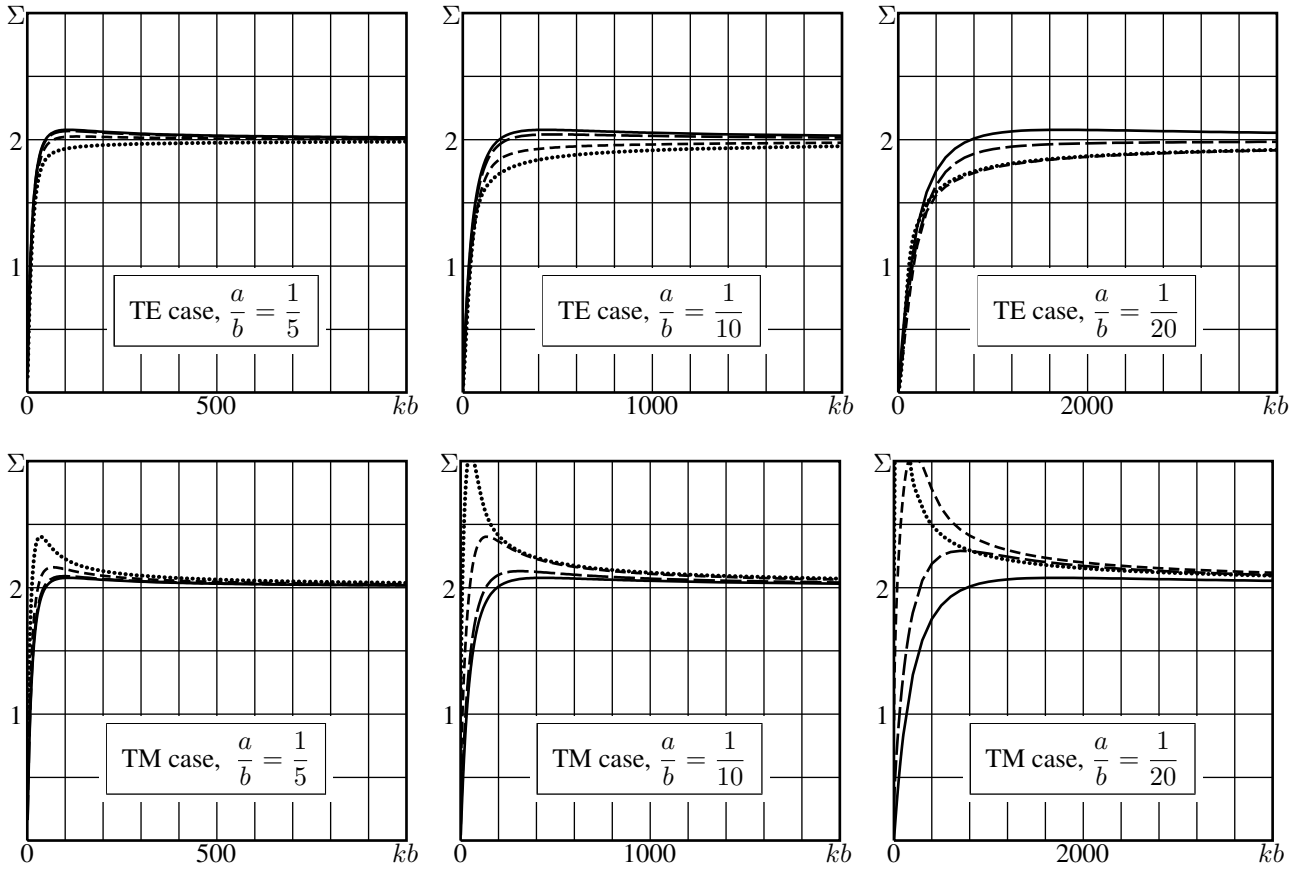

Figure 1: The total scattering cross-section for TE waves (upper) and TM case (lower) on the spheroids with different aspect ratios $a : b$ for the angles of incidence $\vartheta = 0°$ (solid line), $2°$ (long dashes), $5°$ (short dashes) and $10°$ (dots).

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