Beam Domain Formulation for Wave Propagation in Weakly Rough Media

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Abstract

We present a novel strategy for tracking wavefields in weakly fluctuating medium as an extension of the phase-space beam summation method. In this method, which is based on the theory of overcomplete frames, the aperture source is expanded using a windowed Fourier transform frame, thus expressing the radiating field as a sum of beams emerging from a discrete set of points and directions in the source domain. We show that the set of propagating beams constitutes an overcomplete frame in the propagation zone, denoted as the *propagating frame*. This results in a new self consistent formulation for tracking wave fields in fluctuating medium, where the propagating field is described as a sum of beam fields, while the interaction of these beams with the medium is re-expanded using the same set of beams.

1. Introduction - Problem Formulation and Overall Goal

Gaussian beam (GB) summation methods have long been utilized for modeling propagation in complex environments for their unique properties, combining the asymptotically uniform features of the spectral representation with the algorithmic ease of the ray representation (see a review in [1]). The main goal of this work is to introduce a new methodology for tracking wavefields through a weakly scattering medium, using a self-consistent beam summation (BS) approach, namely, a method where the beam-set used to describe the incident field is also used to describe the scattered field. Specifically, the new approach is used here in the context of the ``ultra wide band, phase space, beam summation" method (UWB-PS-BS)[2, 3] (see review in (2)-(4) below).

We consider a time harmonic field $u(\mathbf{r})e^{i\omega t}$ in a 3D space $\mathbf{r} = (\mathbf{x}, z) \quad \mathbf{x} = (x_1, x_2)$. $u_0(\mathbf{x})$, the field in the aperture plane $z = 0$, radiates into the half space $z > 0$, characterized by a homogeneous wavespeed c_0 background with a region $0 \leq z \leq L$ of small perturbations in the refraction index $n(\mathbf{r})$. Assuming weak perturbations, the scattered field may be described by the Born approximation

$$u^s(\mathbf{r}) = k^2 \int d^3r'\, G(\mathbf{r}, \mathbf{r}')\varepsilon(\mathbf{r}')u^i(\mathbf{r}'), \qquad \varepsilon(\mathbf{r}) = n^2(\mathbf{r}) - 1, \qquad k = \omega/c_0, \tag{1}$$

where $u^i(\mathbf{r})$ is the solution in the unperturbed homogeneous medium, and $G(\mathbf{r}, \mathbf{r}')$, the 3D free space Green's function. In particular we would be interested in u^r and u^t, the forward and backward scattered fields at the planes $z = L$ and $z = 0$, respectively (see Figure 1).

The scattering integral (1) may be evaluated either directly, i.e., spatially, or via its spectral (plane-wave) representation. The spatial approach becomes ineffective if the local heterogeneities are known only statistically, typically in terms of the local spectral cross section [4]. On the other hand, the global spectral formulation is suitable *only* if the background is homogeneous *and* the statistical features are uniform, and it also lacks the tractable form of rays, accounting for separate spectral contributions. Both difficulties are addressed effectively by the BS formulation suggested here, as the beam propagators are used to construct the local spectrum (phase-space) representation of the field.

2. The Ultra Wide Band Phase Space Beam Summation (UWB-PS-BS) Method

The UWB-PS-BS decomposes the initial field $u_0(\mathbf{x})$ using a Windowed Fourier Transform (WFT) frame of a Gaussian mother window, in the form [2]

$$u_0(\mathbf{x}) = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}}\psi_{\boldsymbol{\mu}}(\mathbf{x}), \quad \psi_{\boldsymbol{\mu}}(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{m}\bar{x})e^{ik\mathbf{n}\bar{\xi}\cdot(\mathbf{x}-\mathbf{m}\bar{x})}, \quad a_{\boldsymbol{\mu}} = \langle u_0(\mathbf{x}), \varphi_{\boldsymbol{\mu}}(\mathbf{x})\rangle, \tag{2}$$

$\psi_{\boldsymbol{\mu}}$ are the frame elements and $\boldsymbol{\mu} = (\mathbf{m}, \mathbf{n}) = \big((m_1, m_2), (n_1, n_2)\big)$ is a 4-index. The frame functions are localized in the $(\mathbf{x}, \boldsymbol{\xi})$ phase-space around the lattice points $(\mathbf{x}_\mathbf{m}, \xi_\mathbf{n}) = (\mathbf{m}\bar{x}, \mathbf{n}\bar{\xi})$ where $\bar{x}, \bar{\xi}$ are parameters of the expansion (see discussion below). The lattice cell size $\bar{x}\bar{\xi}$ is a measure of the over-completeness: Expressing it as $k\bar{x}\bar{\xi} = 2\pi\nu$ it follows that the formulation is overcomplete for $\nu < 1$, becoming critically complete in the Gabor limit $\nu \uparrow 1$.

In (2), the expansion coefficients $a_{\boldsymbol{\mu}}$ are obtained by projecting $u_0(\mathbf{x})$ onto the "canonical dual frame" $\varphi_{\boldsymbol{\mu}}(\mathbf{x})$ which is also a WFT frame whose elements are given in (2) with the window ψ replaced by the "dual window" φ. Here $\langle f, g\rangle$ is the usual $\mathbb{L}_2$ inner product. In general φ needs to be calculated numerically [2], but if the lattice is highly overcomplete then $\varphi(\mathbf{x}) \approx (\nu/\|\psi\|)^2\psi(\mathbf{x})$. Since the frame is overcomplete, the expansion coefficient set $a_{\boldsymbol{\mu}}$ is not unique but the expression in (2) provides a minimum ℓ_2 norm of this set.

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Substituting the expansion in (2) into the radiation integral, the radiated field in the unperturbed medium is given by

$$u^i(\mathbf{r}) = \sum_{\boldsymbol{\mu}} a_{\boldsymbol{\mu}} B_{\boldsymbol{\mu}}(\mathbf{r}), \qquad B_{\boldsymbol{\mu}}(\mathbf{r}) = \int d^2x' \, F(\mathbf{r}, \mathbf{r}')\psi_{\boldsymbol{\mu}}(\mathbf{x}') = \left(\frac{k}{2\pi}\right)^2 \int d^2\xi \, \tilde{\psi}_{\boldsymbol{\mu}}(\boldsymbol{\xi}) e^{ik(\boldsymbol{\xi}\cdot\mathbf{x} + \sqrt{1-\boldsymbol{\xi}\cdot\boldsymbol{\xi}}z)}, \quad \mathbf{r}' = (\mathbf{x}', 0). \quad (3)$$

$B_{\boldsymbol{\mu}}(\mathbf{r})$ is the radiated field due to each $\psi_{\boldsymbol{\mu}}(\mathbf{x})$ in the aperture plane. The first expression for $B_{\boldsymbol{\mu}}$ is the conventional Kirchhoff integration, with $F(\mathbf{r}, \mathbf{r}') = 2\partial_z' G(\mathbf{r}, \mathbf{r}')$ being the Kirchhoff propagator of the unperturbed medium for radiation from the plane $z' = 0$. The second form for $B_{\boldsymbol{\mu}}$ is the spectral representation, with $\tilde{\psi}(\boldsymbol{\xi}) = \int d^2x \, \psi(\mathbf{x}) e^{-ik\boldsymbol{\xi}\cdot\mathbf{x}}$ being the spectral counterpart of $\psi(\mathbf{x})$. Note that we use the normalized spectral variable $\boldsymbol{\xi} = \mathbf{k}_x/k$ so that $\boldsymbol{\xi}$ is a pure geometrical spectrum parameter, expressing the plane-wave direction via $\theta_{1,2} = \sin^{-1}\xi_{1,2}$, with $\theta_{1,2}$ measured from the $x_{1,2}$ axes, respectively. Since $\psi_{\boldsymbol{\mu}}$ in (2) is localized around $(\mathbf{x_m}, \xi_\mathbf{n})$ it follows that $B_{\boldsymbol{\mu}}$ is a beam field emerging from $\mathbf{x_m}$ in the direction $\xi_\mathbf{n}$.

For UWB applications we choose the parameters $(\bar{x}, \bar{\xi})$ to be frequency independent, thus ensuring that the phase-space beam skeleton is frequency independent. This implies that one needs to track only one set of beams and then use it for all frequencies. Expressing the Gaussian windows in the form $\psi(\mathbf{x}) = e^{-k|\mathbf{x}|^2/2b}$ we find that b is the collimation (Rayleigh) length of the resulting GB propagators $B_{\boldsymbol{\mu}}$. Using the same b for all frequencies implies that the propagation parameters of these GBs are frequency independent even in inhomogeneous media ("iso-diffracting" beams), implying that $B_{\boldsymbol{\mu}}$ need to be calculated only once and then use for all frequencies. For further details the readers are referred to [1-3].

The formulation in (2)-(3) can be cast in a more compact form by using the Kirchhoff operator F

$$u^i(\mathbf{r}) = \langle u_0(\mathbf{r}'), F^*(\mathbf{r}, \mathbf{r}')\rangle, \qquad F(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\mu}} B_{\boldsymbol{\mu}}(\mathbf{r})\varphi^*_{\boldsymbol{\mu}}(\mathbf{x}'), \quad \mathbf{r}' = (\mathbf{x}', 0). \quad (4)$$

where F^* is the complex conjugate of F. In the next section we show that the frame concept, as well as the Kirchhoff propagator representation, can be generalized in a natural way to treat the roughness in the medium.

3. The Propagating Frame

As outlined in the Introduction, our goal is to formulate an UWB-PS-BS representation for field-propagation through a weakly scattering medium. In Sec. 2 we expressed the field radiated from the aperture as a sum of beams emerging from a discrete set of points and directions. Next, it is required to derive a beam-expansion framework for the field interactions with the medium heterogeneities. Toward this goal we show in the present section that the propagating beams set of Sec. 2 constitutes an overcomplete frame at every $z = const$ plane in the propagation zone, denoted here as the *propagating frame*. This new result leads in Sec. 4 to a new self consistent formulation for tracking wave fields in a weakly scattering medium, describing the propagating field as a sum of beam fields, while re-expanding the beams' interaction with the medium using the same beam set.

We consider wave-fields that are confined to the propagating spectrum, and define the Hilbert space $\mathbb{H}_P$

$$\mathbb{H}_P = \left\{ f(\mathbf{x}) \in \mathbb{L}_2 \;\middle|\; \tilde{f}(\boldsymbol{\xi}) = 0 \text{ for } |\boldsymbol{\xi}| \geq \xi_0, \text{ with } \xi_0 < 1 \right\}, \quad (5)$$

where $\xi_0 < 1$ is a parameter. Likewise, we define the frame subset confined to the propagating spectrum

$$\boldsymbol{\mu}_P = \left\{ \boldsymbol{\mu} \in \mathbb{Z}^4 \;\middle|\; \boldsymbol{\mu} = (\mathbf{m}; \mathbf{n}) \text{ with } |\mathbf{n}| < (\xi_0/\bar{\xi}) + n_0 \right\}, \quad (6)$$

where n_0 is a parameter describing the spectral width of ψ. $\mathbb{H}_P$ is limited to ξ_0 rather than one, so that all the relevant frame functions, needed to represent $\mathbb{H}_P$, are in $\boldsymbol{\mu}_P$, without any leakage into the evanescent spectrum.

Theorem 3.1. *Over $\mathbb{H}_P$, the set of propagating frame functions $\Psi_{\boldsymbol{\mu}}(\mathbf{x}; z) = B_{\boldsymbol{\mu}}(\mathbf{r})$, $\boldsymbol{\mu} \in \boldsymbol{\mu}_P$, constitute a frame at any $z = const$ plane. The canonical dual frame $\Phi_{\boldsymbol{\mu}}(\mathbf{x}; z)$ is the canonical dual frame $\varphi_{\boldsymbol{\mu}}(\mathbf{x})$ propagated to z as in (3).*

The proof can be found in [5]. An explanation is the fact that all the frame elements $\Psi_{\boldsymbol{\mu}}$, $\boldsymbol{\mu} \in \boldsymbol{\mu}_P$, can be back propagated to the plane $z = 0$ to reconstruct $\psi_{\boldsymbol{\mu}} \in \boldsymbol{\mu}_P$. Hence the sets are equivalent.

4. Propagating Frame Representation for Propagation in Weakly Rough Media

Using the propagating frame sets $\{\Psi_{\boldsymbol{\mu}}(\mathbf{x}; z), \Phi_{\boldsymbol{\mu}}(\mathbf{x}; z)\}$, one may decompose any propagating field $u(\mathbf{r})$ in the perturbed medium in terms of the propagating beam set in the unperturbed medium, viz

$$u(\mathbf{r}) = \sum_{\boldsymbol{\mu} \in \boldsymbol{\mu}_P} a_{\boldsymbol{\mu}}(z) B_{\boldsymbol{\mu}}(\mathbf{r}), \quad a_{\boldsymbol{\mu}}(z) = \langle u(\mathbf{r}), \Phi_{\boldsymbol{\mu}}(\mathbf{x}; z)\rangle. \quad (7)$$

Specifically, one may readily show that if $u_0 \in \mathbb{H}_P$ then the unperturbed field $u^i(\mathbf{r})$ satisfies $a_{\boldsymbol{\mu}}(z) = \langle u^i(\mathbf{r}), \Phi_{\boldsymbol{\mu}}(\mathbf{x}; z)\rangle = \langle u_0(\mathbf{x}), \varphi_{\boldsymbol{\mu}}(\mathbf{x})\rangle = a_{\boldsymbol{\mu}}(0)$.

Applying (7) in the perturbed medium, it is therefore sufficient to track how the coefficients $a_{\boldsymbol{\mu}}(z)$ are transformed as a function of z. Accounting for the problem linearity in the initial field, it follows that the coefficients transformation has the generic form

$$a_{\boldsymbol{\mu}}(z) = \sum_{\boldsymbol{\mu}'} V_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z) a_{\boldsymbol{\mu}'}(0), \quad V_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z) = \delta_{\boldsymbol{\mu}\boldsymbol{\mu}'} + V^s_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z). \tag{8}$$

$\delta_{\boldsymbol{\mu}\boldsymbol{\mu}'}$ is the Kronecker delta, while $V^s_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z)$ represent beam to beam scattering caused by the medium heterogeneities as illustrated by figure 2. Thus, the propagating frame representation offers a direct and compact way to account for beam to beam scattering effects within the framework of the UWB-PS-BS. This result can be expressed more generally in the context of the Kirchhoff propagator operator in a general medium.

Theorem 4.1. $F_P(\mathbf{r}, \mathbf{r}')$ *- the Kirchhoff propagator operator in a general medium, from* $\mathbb{H}_P$ *in* $z = 0$ *onto* $\mathbb{H}_P$ *in* z*, has the general form [5]*

$$F_P(\mathbf{r}, \mathbf{r}') = \sum_{\boldsymbol{\mu}\boldsymbol{\mu}'} V_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z) B_{\boldsymbol{\mu}}(\mathbf{r}) \varphi^*_{\boldsymbol{\mu}'}(\mathbf{x}') \quad \boldsymbol{\mu}, \boldsymbol{\mu}' \in \boldsymbol{\mu}_P \tag{9}$$

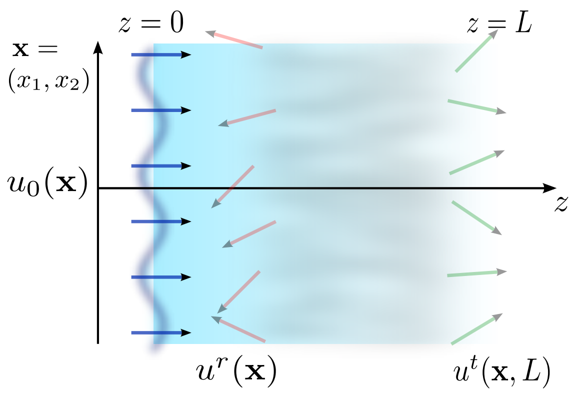

Figure 1: Incident, forward and backward scattered fields

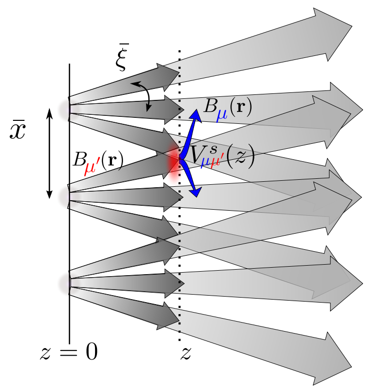

Figure 2: Beam to beam coupling via $V_{\boldsymbol{\mu}\boldsymbol{\mu}'}$

5. Calculation of $V_{\boldsymbol{\mu}\boldsymbol{\mu}'}$ in the Born Approximation

Substituting the propagating frame representation for u^i and $G(\mathbf{r}', \mathbf{r}')$ into (1), one obtains an integral representation for $V^s_{\boldsymbol{\mu}\boldsymbol{\mu}'}$ within the Born approximation, in the form of beam-to-beam correlation integrals. The complete derivation will appear in [5]. In the following example we apply the propagating frame decomposition to calculate the Born approximation for the forward scattered field in a fluctuating medium for a 2D (x, z) GB propagating along the z-axis at two frequencies such that $\lambda_1 = 1$ and $\lambda_2 = 3/2$. The medium fluctuations are confined in the range $z \in (0, L = 10)$ and the have the form

$$\varepsilon(\mathbf{r}) = \mathcal{N}(\mathbf{r}) \sin(2\pi\mathbf{x}/\Lambda), \quad \Lambda = 3, \tag{10}$$

where $\mathcal{N}(\mathbf{r})$ is a sample of a 2D Gaussian process with moments $\langle\mathcal{N}(\mathbf{r})\rangle = 0$ and $\langle\mathcal{N}(\mathbf{r})\mathcal{N}(\mathbf{r}')\rangle \propto e^{|\mathbf{r}-\mathbf{r}'|/\ell}$ with $\ell = 10$. A typical realization is plotted in Fig. 3 using the units of λ_1. Note the different scales in the x, z axes.

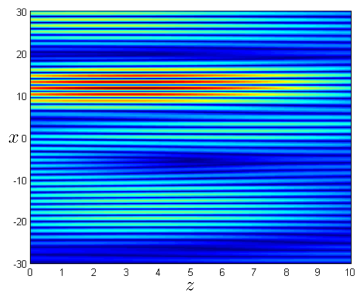

Figure 3: The medium realization

The phase space beam expansion parameters were chosen such that the beam collimation length was $b = 10$, and the overcompleteness parameter at the highest frequency $\lambda_1 = 1$ was $\nu = 0.25$. Following the rules in [2] for choosing the phase space parameters the parameters in (2) are $\bar{x} = \sqrt{2.5}$ and $\bar{\xi} = \sqrt{2.5}/b$. The initial field is the $\boldsymbol{\mu} = (0,0)$ frame element.

Figs. 4 and 5 depict the Born approximation of the scattered fields for $\lambda_{1,2}$, respectively, along with phase-space maps of the absolute value of $a_{\boldsymbol{\mu}}(z) = \sum_{\boldsymbol{\mu}'} V^s_{\boldsymbol{\mu}\boldsymbol{\mu}'}(z) a_{\boldsymbol{\mu}'}(0)$ at several z planes. Comparing Figs. 4(a) and 5(a) one may see that the transversal medium fluctuation in (11) have caused a beam splitting which is larger at the lower frequency λ_2. From the phase space maps one may readily discern that the coefficients dynamics capture the dynamics of the scattered field. Specifically, one observes the beam splitting at angles $\xi \sim 2\bar{\xi}$ and $\xi \sim 3\bar{\xi}$ for the $\lambda_{1,2}$ cases, respectively, and the separation of the beam maxima with increasing range. In addition, the spreading of the phase space spots as a function of the range describe the beam diffusion due to the random scattering along the propagation axis.

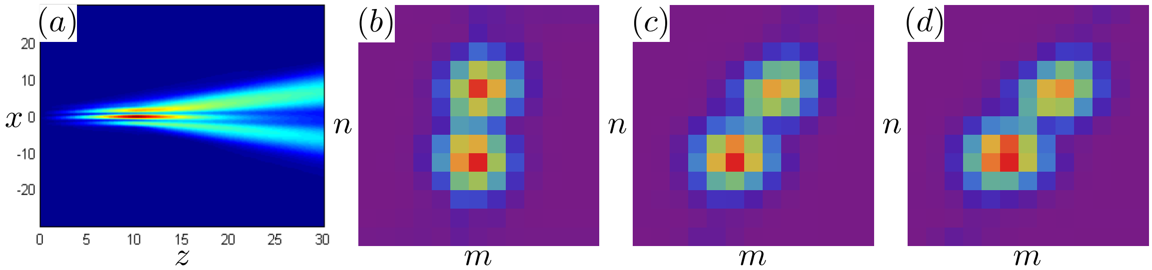

Figure 4: $\lambda_1 = 1$ (a) $|u^s(x,z)|$ (Born). (b,c,d) $|a_{\boldsymbol{\mu}}(z)|@\ z = 2, 4, 18,$ respectively. m, n are the coefficient coordinates of the phase-space lattice, with $(m,n) = (0,0)$ at the center.

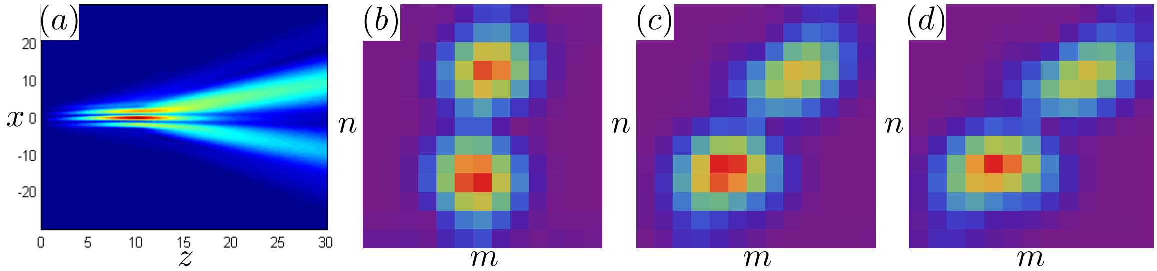

Figure 5: As in figure 4, but for $\lambda_2 = 3/2$

6. Conclusions and Future Work

We presented an extension of the UWB-PS-BS method for treating propagation through rough media. By proving that the set of propagating beams can be used as frames, we obtained an effective method for incorporating the scattered field into the general propagation problem. Future work will include an extension to stochastic field theory for random media where the beam scattering coefficients $V^s_{\boldsymbol{\mu}\boldsymbol{\mu}'}$ in (8) are stochastic constituents that are related to the statistical properties of the medium.

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8. References

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