Skewon-Axion Medium as Boundary Material

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Abstract

Skewon-axion medium is defined by 16 free parameters out of 36. A plane wave in such a medium is not restricted by a dispersion equation. Applying plane-wave equations in four-dimensional formalism, it is shown that an interface of a skewon-axion medium can be defined to produce a boundary condition which generalizes the DB and soft-and-hard boundary conditions to SHDB conditions. The SHDB boundary is self-dual, invariant in any three-parameter duality transformation.

1. Introduction

Applying the four-dimensional differential-form formalism of [1,2], the Maxwell equations outside the sources can be compactly expressed as

$$\mathbf{d}\wedge\mathbf{\Phi}=0, \qquad \mathbf{d}\wedge\mathbf{\Psi}=0, \tag{1}$$

where, $\mathbf{\Phi}$ and $\mathbf{\Psi}$ denote the electromagnetic two-forms

$$\mathbf{\Psi}=\mathbf{D}-\mathbf{H}\wedge\boldsymbol{\varepsilon}_4, \qquad \mathbf{\Phi}=\mathbf{B}+\mathbf{E}\wedge\boldsymbol{\varepsilon}_4. \tag{2}$$

Here, $\boldsymbol{\varepsilon}_4=\mathbf{d}\tau=c\mathbf{d}t$ is the temporal basis one-form and $\mathbf{E},\mathbf{H},\mathbf{D},\mathbf{B}$ are 3D (spatial) one-form and two-form components. The medium equation

$$\mathbf{\Psi}=\overline{\overline{\mathsf{M}}}|\mathbf{\Phi}, \tag{3}$$

involves the medium bidyadic $\overline{\overline{\mathsf{M}}}$ (6×6 matrix), which corresponds to four dyadics $\overline{\overline{\alpha}},\overline{\overline{\epsilon}}',\overline{\overline{\mu}}^{-1},\overline{\overline{\beta}}$ in the equation

$$\begin{pmatrix}\mathbf{D}\\ \mathbf{H}\end{pmatrix}=\begin{pmatrix}\overline{\overline{\alpha}} & \overline{\overline{\epsilon}}'\\ \overline{\overline{\mu}}^{-1} & \overline{\overline{\beta}}\end{pmatrix}|\begin{pmatrix}\mathbf{B}\\ \mathbf{E}\end{pmatrix} \tag{4}$$

between the 3D field quantities. Details on the formalism applied here can be found in [2].

There is a natural decomposition of the medium bidyadic $\overline{\overline{\mathsf{M}}}$ in three components by Hehl and Obukhov in [3] as

$$\overline{\overline{\mathsf{M}}}=\overline{\overline{\mathsf{M}}}_1+\overline{\overline{\mathsf{M}}}_2+\overline{\overline{\mathsf{M}}}_3 \tag{5}$$

respectively called the principal, skewon and axion parts. The axion part is a scalar multiple of the unit bidyadic expressed as

$$\overline{\overline{\mathsf{M}}}_3=M\overline{\overline{\mathsf{I}}}^{(2)T}, \tag{6}$$

while both $\overline{\overline{\mathsf{M}}}_1$ and $\overline{\overline{\mathsf{M}}}_2$ are trace-free bidyadics. The components $\overline{\overline{\mathsf{M}}}_1$ and $\overline{\overline{\mathsf{M}}}_2$ can be defined so that the bidyadics contracted by a quadrivector $\mathbf{e}_N=\mathbf{e}_{1234}$ as $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}_1$ and $\mathbf{e}_N\lfloor\overline{\overline{\mathsf{M}}}_2$ are respectively symmetric and antisymmetric. The total number of 36 parameters is distributed to the three components so that the principal part $\overline{\overline{\mathsf{M}}}_1$, corresponding to a trace-free symmetric 6×6 matrix has 20 parameters, the skewon part $\overline{\overline{\mathsf{M}}}_2$, corresponding to an antisymmetric 6×6 matrix, has 15 parameters, and the axion part $\overline{\overline{\mathsf{M}}}_3$, has 1 parameter. The principal component $\overline{\overline{\mathsf{M}}}_1$ is responsible for the "ordinary" properties of the medium while the skewon component $\overline{\overline{\mathsf{M}}}_2$ gives rise to such effects as gyrotropy and chirality. A medium consisting only of its axion parameter, $\overline{\overline{\mathsf{M}}} = \overline{\overline{\mathsf{M}}}_3$ in (6), corresponds to the medium conditions

$$\mathbf{D} = M\mathbf{B}, \qquad \mathbf{H} = -M\mathbf{E}, \tag{7}$$

and has been called PEMC, perfect electromagnetic conductor, because it is a generalization of both PMC ($M = 0$) and PEC ($1/M = 0$), [4]. A skewon-axion medium lacks the principal component and involves 16 parameters. Such a medium bidyadic can be expressed in terms of a dyadic $\overline{\overline{\mathsf{B}}}$, corresponding to a 4×4 matrix, as [3,5]

$$\overline{\overline{\mathsf{M}}} = (\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge \overline{\overline{\mathsf{I}}})^T. \tag{8}$$

For Gibbsian field vectors denoted by $\mathbf{E}_g, \mathbf{H}_g, \mathbf{D}_g, \mathbf{B}_g$ the skewon-axion medium conditions can be expressed in the form

$$\mathbf{D}_g = (\overline{\overline{\mathsf{A}}} - \mathrm{tr}\overline{\overline{\mathsf{A}}}\ \overline{\overline{\mathsf{I}}}) \cdot \mathbf{B}_g + \mathbf{a} \times \mathbf{E}_g, \tag{9}$$

$$\mathbf{H}_g = \mathbf{b} \times \mathbf{B}_g + (\overline{\overline{\mathsf{A}}}{}^T + \alpha\overline{\overline{\mathsf{I}}}) \cdot \mathbf{E}_g, \tag{10}$$

where $\mathbf{a}, \mathbf{b}$ are two Gibbsian vectors, $\overline{\overline{\mathsf{A}}}$ is a Gibbsian dyadic and α is a scalar, making $3 + 3 + 9 + 1 = 16$ parameters. It is not possible to express the medium conditions in the "engineering" form $\mathbf{D}_g = \overline{\overline{\epsilon}}_g \cdot \mathbf{E}_g + \overline{\overline{\xi}}_g \cdot \mathbf{H}_g$ and $\mathbf{B}_g = \overline{\overline{\zeta}}_g \cdot \mathbf{E}_g + \overline{\overline{\mu}}_g \cdot \mathbf{H}_g$ because $\overline{\overline{\mu}}_g$ does not exist.

2. Plane Wave in a Skewon-Axion Medium

Considering a time-harmonic plane wave $\mathbf{E}_g(\mathbf{r}) = \mathbf{E}_g \exp(-j\mathbf{k}.\mathbf{r})$ in a skewon-axion medium defined by the Gibbsian conditions (9), (10), from the Maxwell equations one can obtain the equation for the field $\mathbf{E}_g$

$$\mathbf{q}(\mathbf{p}) \cdot \mathbf{E}_g = 0, \qquad \mathbf{q}(\mathbf{p}) = (\alpha - \mathbf{b} \cdot \mathbf{p})\mathbf{p} + \mathbf{p} \cdot \overline{\overline{\mathsf{A}}} + \mathbf{a}, \quad \mathbf{p} = \mathbf{k}/\omega, \tag{11}$$

which does not yield any dispersion equation for $\mathbf{k}$. Thus, one can choose any $\mathbf{k}$ vector for a plane wave in a skewon-axion medium.

In 4D formalism, the plane wave obeys the form $\mathbf{\Phi}(\mathbf{x}) = \mathbf{\Phi}\exp(\boldsymbol{\nu}|\mathbf{x})$, $\mathbf{\Psi}(\mathbf{x}) = \mathbf{\Psi}\exp(\boldsymbol{\nu}|\mathbf{x})$ where $\mathbf{x}$ is the space-time four-vector and $\boldsymbol{\nu}$ the wave one-form which corresponds to the $\mathbf{k}$ vector. From the Maxwell equation we obtain

$$\boldsymbol{\nu} \wedge \mathbf{\Phi} = 0, \quad \Rightarrow \quad \mathbf{\Phi} = \boldsymbol{\nu} \wedge \boldsymbol{\phi}, \tag{12}$$

where $\boldsymbol{\phi}$ is a potential one-form. The second Maxwell equation $\boldsymbol{\nu} \wedge \mathbf{\Psi} = 0$ for the skewon-axion medium can be expanded as

$$\boldsymbol{\nu} \wedge \mathbf{\Psi} = \boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}{}^T{}^\wedge_\wedge \overline{\overline{\mathsf{I}}}{}^T)|(\boldsymbol{\nu} \wedge \boldsymbol{\phi}) \tag{13}$$

$$= \boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}{}^T|\boldsymbol{\nu}) \wedge \boldsymbol{\phi} = 0, \tag{14}$$

whence the three one-forms must be linearly dependent. Thus, the potential can be expressed as

$$\boldsymbol{\phi} = \alpha\boldsymbol{\nu} + \beta\overline{\overline{\mathsf{B}}}{}^T|\boldsymbol{\nu}, \tag{15}$$

in terms of some scalars α, β. Thus, the field two-forms must have the form

$$\mathbf{\Phi} = \boldsymbol{\nu} \wedge \boldsymbol{\phi} = \beta\boldsymbol{\nu} \wedge \overline{\overline{\mathsf{B}}}{}^T|\boldsymbol{\nu}, \tag{16}$$

$$\begin{aligned} \mathbf{\Psi} &= (\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge \overline{\overline{\mathsf{I}}})^T|\mathbf{\Phi} = \beta(\overline{\overline{\mathsf{B}}}{}^\wedge_\wedge \overline{\overline{\mathsf{I}}})^T|(\boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}{}^T|\boldsymbol{\nu})) \\ &= \beta\boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}{}^{2T}|\boldsymbol{\nu}). \end{aligned} \tag{17}$$

The relations of the two field two-forms in a skewon-axion medium depends on the dyadic $\overline{\overline{\mathsf{B}}}$ defining the medium. In the general case, we can expand the dyadic $\overline{\overline{\mathsf{B}}} = \sum B_{ij}\mathbf{e}_i\boldsymbol{\varepsilon}_j$ where $\mathbf{e}_i$ makes a basis of vectors and $\boldsymbol{\varepsilon}_j$ a basis of one-forms. One can show that there is no restriction for the choice of the wave one-form $\boldsymbol{\nu}$ of a plane wave in a skewon-axion medium.

3. Boundary Conditions

For some special dyadics $\overline{\overline{\mathsf{B}}}$ the field two-forms may be restricted by the skewon-axion medium so that fields at the interface of such a medium are likewise restricted. This means that it is possible to obtain meaningful boundary conditions at the interface of a skewon-axion medium. Assuming a planar interface defined by $\boldsymbol{\varepsilon}_3|\mathbf{x} = 0$ which corresponds to the Gibbsian condition $\mathbf{e}_3 \cdot \mathbf{r} = 0$ with vector $\mathbf{e}_3$ normal to the interface, continuity of the fields $\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\Phi}$ and $\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\Psi}$ at the interface is required. Separating spatial and temporal parts, these are equivalent to requiring continuity at the interface of $\boldsymbol{\varepsilon}_3 \wedge \mathbf{B}$ and $\boldsymbol{\varepsilon}_3 \wedge \mathbf{D}$ on one hand and of $\boldsymbol{\varepsilon}_3 \wedge \mathbf{E}$ and $\boldsymbol{\varepsilon}_3 \wedge \mathbf{H}$ on other hand. For the respective Gibbsian fields they correspond to continuity of the normal component of $\mathbf{B}_g$ and $\mathbf{D}_g$ and the tangential component of $\mathbf{E}_g$ and $\mathbf{H}_g$.

As an example, let us consider dyadic $\overline{\overline{\mathsf{B}}}$ of the simple form

$$\begin{aligned}\overline{\overline{\mathsf{B}}} &= B\overline{\overline{\mathsf{I}}} + \mathbf{a}\boldsymbol{\alpha}, && (18)\\ \overline{\overline{\mathsf{B}}}^2 &= B^2\overline{\overline{\mathsf{I}}} + (2B + \mathbf{a}|\boldsymbol{\alpha})\mathbf{a}\boldsymbol{\alpha}, && (19)\end{aligned}$$

where $\boldsymbol{\alpha}$ is a one-form and $\mathbf{a}$ is a vector. In this case the field two-forms (16), (17) in the medium can be expressed as

$$\begin{aligned}\boldsymbol{\Phi} &= \beta\boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}^T|\boldsymbol{\nu}) = \beta(\mathbf{a}|\boldsymbol{\nu})\boldsymbol{\nu} \wedge \boldsymbol{\alpha} && (20)\\ \boldsymbol{\Psi} &= \beta\boldsymbol{\nu} \wedge (\overline{\overline{\mathsf{B}}}^{(2T}|\boldsymbol{\nu}) = \beta(2B + \mathbf{a}|\boldsymbol{\alpha})(\mathbf{a}|\boldsymbol{\nu})\boldsymbol{\nu} \wedge \boldsymbol{\alpha}. && (21)\end{aligned}$$

Thus, in such a medium, the fields of any plane wave are restricted by

$$\boldsymbol{\alpha} \wedge \boldsymbol{\Phi} = 0, \qquad \boldsymbol{\alpha} \wedge \boldsymbol{\Psi} = 0. \tag{22}$$

From the continuity condition it follows that fields at the interface satisfy the conditions

$$\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\Phi} = 0, \qquad \boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\Psi} = 0. \tag{23}$$

Without sacrificing the generality, we can expand

$$\boldsymbol{\alpha} = \alpha_1\boldsymbol{\varepsilon}_1 + \alpha_3\boldsymbol{\varepsilon}_3 + \alpha_4\boldsymbol{\varepsilon}_4, \tag{24}$$

whence the conditions (23) become

$$\begin{aligned}\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\Phi} &= \boldsymbol{\varepsilon}_3 \wedge (\alpha_1\boldsymbol{\varepsilon}_1 + \alpha_4\boldsymbol{\varepsilon}_4) \wedge \boldsymbol{\Phi}\\ &= \boldsymbol{\varepsilon}_3 \wedge (\alpha_1\boldsymbol{\varepsilon}_1 \wedge \mathbf{E} \wedge \boldsymbol{\varepsilon}_4 + \alpha_4\boldsymbol{\varepsilon}_4 \wedge \mathbf{B}) = 0, && (25)\\ \boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\alpha} \wedge \boldsymbol{\Psi} &= \boldsymbol{\varepsilon}_3 \wedge (\alpha_1\boldsymbol{\varepsilon}_1 + \alpha_4\boldsymbol{\varepsilon}_4) \wedge \boldsymbol{\Psi}\\ &= \boldsymbol{\varepsilon}_3 \wedge (-\alpha_1\boldsymbol{\varepsilon}_1 \wedge \mathbf{H} \wedge \boldsymbol{\varepsilon}_4 + \alpha_4\boldsymbol{\varepsilon}_4 \wedge \mathbf{D}) = 0. && (26)\end{aligned}$$

These are equivalent to the spatial conditions

$$\begin{aligned}\mathbf{f}(\mathbf{E}, \mathbf{B}) &= \alpha_1\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\varepsilon}_1 \wedge \mathbf{E} + \alpha_4\boldsymbol{\varepsilon}_3 \wedge \mathbf{B} = 0, && (27)\\ \mathbf{f}(-\mathbf{H}, \mathbf{D}) &= -\alpha_1\boldsymbol{\varepsilon}_3 \wedge \boldsymbol{\varepsilon}_1 \wedge \mathbf{H} + \alpha_4\boldsymbol{\varepsilon}_3 \wedge \mathbf{D} = 0, && (28)\end{aligned}$$

which correspond to the following conditions for the Gibbsian field vectors,

$$\begin{aligned}\alpha_1\mathbf{e}_2 \cdot \mathbf{E}_g + \alpha_4\mathbf{e}_3 \cdot \mathbf{B}_g &= 0, && (29)\\ -\alpha_1\mathbf{e}_2 \cdot \mathbf{H}_g + \alpha_4\mathbf{e}_3 \cdot \mathbf{D}_g &= 0. && (30)\end{aligned}$$

Since for $\alpha_1 = 0$ (29) and (30) coincide with the DB boundary conditions [6] and, for $\alpha_4 = 0$, with the soft-and-hard (SH) conditions [7], they can be dubbed as SHDB conditions [8].

4. Discussion and Conclusion

Because the SHDB conditions are obtained for any plane wave in the skewon-axion medium, due to linearity, they are satisfied for any sum or integral of plane waves, i.e., for any fields in the medium. To obtain an SHDB boundary surface, the amount of skewon-axion medium below the interface can be small, actually a thin layer is enough. Also, since the boundary condition is local, the boundary surface can be more general than a plane. This makes it possible to define an SHDB metasurface by a suitable materialization of the skewon-axion medium.

As an additional property, one can show that the SHDB conditions (29) and (30) are self dual, i.e., they are invariant in any three-parameter duality transformation of the form [2]

$$\begin{pmatrix} Z_d\mathbf{\Psi}_d \\ \mathbf{\Phi}_d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} Z\mathbf{\Psi} \\ \mathbf{\Phi} \end{pmatrix}. \tag{31}$$

In fact, from the representation

$$\begin{pmatrix} \mathbf{f}(\mathbf{E}_d,\mathbf{B}_d) \\ \mathbf{f}(-\mathbf{H}_d,\mathbf{D}_d) \end{pmatrix} = \begin{pmatrix} \cos\theta & -Z\sin\theta \\ (1/Z_d)\sin\theta & (Z/Z_d)\cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{f}(\mathbf{E},\mathbf{B}) \\ \mathbf{f}(-\mathbf{H},\mathbf{D}) \end{pmatrix} \tag{32}$$

we obtain $\mathbf{f}(\mathbf{E}_d,\mathbf{B}_d) = 0$ and $\mathbf{f}(-\mathbf{H}_d,\mathbf{D}_d) = 0$. The same property has been known to be valid for the two special cases, the DB boundary and the soft-and-hard (SH) boundary. Thus, for example, an object with an SHDB boundary and suitable symmetry has zero backscattering, i.e., it is invisible for the radar [9].

As a conclusion, Skewon-axion medium appears to be a useful concept in defining important boundary conditions like the PEC, PMC, PEMC, SH, DB and SHDB conditions.

5. References

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