On whistler-mode wave scattering on small scale density irregularities

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Abstract

This work deals with the problem of whistler exit to the ground. After propagation in the magnetosphere in non-ducted mode, whistler waves become quasi-electrostatic and should experience lower hybrid resonance reflection. To explain wave exit to the ground, we follow up the idea of wave scattering on small scale irregularities in the upper ionosphere. Interaction with this irregularities excites harmonics inside the penetration cone. Using Green function method, we derive an integral equation for the wave field containing all modes possible for given frequency, solve this equation in Born approximation, and obtain an expression for energy attenuation due to scattering.

1 Introduction

Recently, many papers on experimental observations of whistler waves induced by lightning discharges have been published. Still the problem of whistler exit to the ground is not completely understood. Numerical calculations in the framework of geometrical optics show that in most cases the wave vector becomes very large at high latitudes in the hemisphere opposite to the wave source. If, in addition, wave frequency is below the local lower hybrid resonance (LHR) frequency, the wave cannot propagate and experiences LHR reflection. Hence we would not observe so many whistlers on the ground, if there were no mechanisms for the wave to get through this region without being reflected. There are two generally accepted ways to overcome this difficulty. The first is to assume that the wave propagates in the magnetosphere in a duct [1,2]. Such ducted wave is preserved from enormous increasing of its wave vector, thus avoiding LHR reflection. Second mechanism is the wave scattering on small scale irregularities of electron density in magnetospheric plasma [3]. Due to these irregularities wave field becomes enriched with different spatial harmonics which propagate without being reflected. But even having assumed ducted propagation in the magnetosphere, we will meet similar problem closer to the ground. A duct ends at the upper layers of the ionosphere and for intermediate latitudes the angle of incidence, namely, the angle between wave vector and vertical to the ground is quite large. From Maxwell's equations it follows that upon exit from the ionosphere to the atmosphere only waves with small enough angle of incidence can convert into free space electromagnetic mode which reaches the ground (see [1]). Otherwise wave will be totally reflected from the atmosphere. Hence, in order to explain whistlers observed on the ground some mechanism is needed for exciting new, almost vertical, spatial harmonics in the wave field. And again a good candidate for such mechanism is scattering on small scale fluctuations of plasma density. For the ionosphere, it could be well known F-spread. Accordingly, the necessity of developing of scattering theory for whistler waves arises. There is a number of works dealing with the problem of wave propagation in random medium and scattering on fluctuations. Budden was the first who considered such scattering as an explanation of whistler observations. Next Antani and Kaup [4] developed a comprehensive theory of whistler wave propagation through the boundless random medium. They obtained differential equation for an averaged field and calculated some power properties of the wave. Simonoch and Yeh [5] studied the scattering of ordinary and extraordinary waves on the localized region of fluctuations in the Born approximation. They obtained interesting results concerning scattering cross-section and considered cross-mode scattering. Important works on the problem were performed by Shklyar et. al.[6,7]. The first paper deals with a general approach to non-stationary propagation of VLF wave packet through medium with small scale irregularities and excitation of electrostatic waves by longitudinal waves due to scattering on these irregularities. The second paper proposes a new method describing scattering of whistler waves with $\theta \nrightarrow \pi/2$, where θ is the angle between external magnetic field and wave vector, on density fluctuations. With this method, the authors succeeded in calculation of energy attenuation of incident signal due to scattering.

One of the main objectives of the present work is, in a sense, to combine these two works, i.e. to propose

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a theory allowing all wave vectors of incident whistler wave to be considered. Besides, we aim at presenting an integral equation convenient for exceeding the limits of Born approximation, which was used in previous works. Although Born approximation is, strictly speaking, invalid for the case of strong enough fluctuations, it gives a good qualitative approach to the problem. Nevertheless, it is useful to estimate the next terms in the Born series in order to get an idea about quantitative specifications of the process.

2 Main equations

We will describe monochromatic electromagnetic field by complex vector potential $\mathbf{A}$ using Coulomb gauge. In cold magnetized plasma vector potential obeys Maxwell's equations

$$\mathrm{rot}\,\mathrm{rot}\,\mathbf{A} = \frac{\omega^2}{c^2}\,\hat{\varepsilon}\,\mathbf{A} \tag{1}$$

with the dielectric tensor

$$\hat{\varepsilon} \equiv \varepsilon_{ij} = \begin{pmatrix} \varepsilon_1 & i\varepsilon_2 & 0 \\ -i\varepsilon_2 & \varepsilon_1 & 0 \\ 0 & 0 & \varepsilon_3 \end{pmatrix} \tag{2}$$

Cartesian coordinate system (x, y, z) is chosen so as z-axis is directed along external magnetic field $\mathbf{B}_0$ which is assumed to be constant, and wave vector lies in the (x, z)-plane. Hence, given the dielectric tensor (2) independent of y, one can search for solutions of Maxwell equations (1) that depend only on x and z, so that the problem becomes two dimensional. In low frequency whistler-mode range dielectric tensor components can be written in a convenient form using notion introduced in [6], namely

$$\frac{\varepsilon_1}{c^2} = \frac{\omega_p^2+\omega_c^2}{c^2\,\omega_c^2} - \frac{\omega_p^2+\omega_c^2}{c^2\,\omega_c^2}\frac{\omega_{LH}^2}{\omega^2} \equiv \alpha - \frac{\beta}{\omega^2}\ ; \quad \frac{\varepsilon_2}{c^2} = -\frac{\omega_p^2}{c^2\,\omega\omega_c} \equiv -\frac{\gamma}{\omega}\ ; \quad \frac{\varepsilon_3}{c^2} = -\frac{\omega_p^2}{c^2\,\omega^2} \equiv -\frac{\eta}{\omega^2}\,, \tag{3}$$

with plasma frequency $\omega_p^2 = 4\pi n_e e^2/m$, cyclotron frequency $\omega_c = eB_0/mc$ and ω_{LH} - lower hybrid frequency. We assume that the medium is homogeneous on average and inhomogeneity arises due to irregularities of electron density. It implies that any quantity from α, β, γ can be divided into two parts: a regular part and a small random part. For example, for α one has $\alpha(x, z) = \alpha_0 + \delta\alpha(x, z)$ where $\delta\alpha \ll \alpha_0$. Let us rewrite (1) taking into account this division

$$\hat{\mathbf{M}}\mathbf{A} = \mathbf{R}\mathbf{A}, \tag{4}$$

where $\hat{M}$ is a regular matrix differential operator, which contains left-hand side and regular part of right-hand side of (1), and $\mathbf{R}$ is a random matrix containing irregular part of right-hand side of (1). Besides, we can neglect small terms in equation for $\mathbf{A}_y$ from (1) having used the fact that for whistler waves $k^2 \gg (\omega_p^2/c^2)(\omega/\omega_c)$, k is an absolute value of typical wave vector. Proceeding to analysis of (4), let us first study the left-hand side of this equation. It is a system of three differential equations with constant coefficients and one can look for solution in the form of plane wave $\mathbf{A}_j \sim \exp(i\,p\,x + i\,q\,z)$. It gives us dispersion equation, which relates components p and q of wave vector. For whistler frequency range it is more convenient to regard q as a function of p. So let us denote solutions of dispersion equation as $(q_r(p),\, -q_r(p),\, i\,q_i(p),\, -i\,q_i(p))$ where $q_r,\, q_i > 0$. Having obtained wave vectors we can find out polarization vectors of the wave which are denoted as $(\boldsymbol{a}^{(1)}(p),\, \boldsymbol{a}^{(2)}(p))$ (hereinafter indexes 1 and 2 stand for vectors of propagating mode, with real wave vector). To state the problem completely, we should specify boundary conditions. We will assume that the region of fluctuation is confined in z direction, but can be infinitely large in x direction. Consequently, the boundary conditions for (1) should be set at z boundaries. Thus considering the problem of wave scattering on the layer (finite in z direction and arbitrary in x direction) of fluctuations we have the following boundary conditions

$$\begin{gathered}\mathbf{A}(x, z \to \infty) \longrightarrow \mathbf{A}^{inc}(x, z) + \int \frac{dp}{2\pi}\, r(p)\boldsymbol{a}^{(1)}(p)e^{i\,q_r(p)\,z + i\,p\,x}, \\ \mathbf{A}(x, z \to -\infty) \longrightarrow \int \frac{dp}{2\pi}\, t(p)\boldsymbol{a}^{(2)}(p)e^{-i\,q_r(p)\,z + i\,p\,x},\end{gathered} \tag{5}$$

where $\mathbf{A}^{inc}$ is an incident wave having the form $\mathbf{A}^{inc} = \int dp/2\pi C^{inc}(p)\boldsymbol{a}^{(2)}(p)\exp(-i\,q_r(p)\,z + i\,p\,x)$, and $t(p)$ and $r(p)$ are transmission and reflection coefficients, respectively. These coefficients have simple physical meaning. Indeed, one can easily prove that solutions of equation (4) obey energy conservation law which can be written in the form

$$\int \frac{dp}{2\pi}\left(v_1(p)\,|r(p)|^2 + v_2(p)|C^{inc}(p)|^2\right) = \int \frac{dp}{2\pi} v_2(p)\,|t(p)|^2, \tag{6}$$

where $v_\lambda(p)$ are group velocities, $\lambda = 1,\,2$. So these coefficients are the amplitudes of reflected and transmitted waves.

3 Green function

Considering weak fluctuations we are able to solve equation (4) using the method of successive approximations. It is more convenient to proceed to the integral equation from differential equation (4). In order to do this we need a Green function, or, here, Green matrix defined as a solution of following equation

$$\hat{\mathbf{M}}\mathbf{G}(x,x',z,z') = \mathbf{1}\delta(x-x')\delta(z-z'), \tag{7}$$

where δ stands for Dirac delta function. This equation is easily solved in Fourier representation

$$\mathbf{G}(p,q) = \mathbf{M}^{(-1)}(p,q) = \frac{1}{det\mathbf{M}}\mathbf{L}(p,q)\,,\ det\mathbf{M} = -\eta_0\,(q-q_r(p))(q+q_r(p))(q-i\,q_i(p))(q+i\,q_i(p)), \tag{8}$$

where $\mathbf{L}(p,q)$ is the transposed matrix of cofactors. Integration on q in inverse Fourier transform for Green function is straightforward. Components of $\mathbf{L}(p,q)$, except one of them, behave like $\lesssim q^3$ when $q \to \infty$ so one can apply Jordan lemma to these integrals. But $\mathbf{L}_{33}(p,q)$ component is proportional to q^4, so in order to use Jordan lemma we need to integrate it separately. Integration over q also implies choosing a way to round the poles. As usual it is determined by boundary conditions (5). After a few transformations one can easily obtain following integral equation

$$\mathbf{A}(x,z) = \int \frac{dp}{2\pi} C^{inc}(p)\boldsymbol{a}^{(2)}(p)\exp(-i\,q_r(p)\,z + i\,p\,x) + \int d^2r'\mathbf{G}(x-x',z-z')\,\mathbf{R}(x',z')\,\mathbf{A}(x',z'), \tag{9}$$

with

$$\begin{aligned}\mathbf{G}(x,z) = -\frac{i}{\eta_0}\int \frac{dp}{2\pi}\Big[&\frac{\mathbf{L}(p,q_r(p))}{2\,q_r(p)(q_r^2+q_i^2)}e^{i\,q_r\,z}\,\theta(z) + \frac{\mathbf{L}(p,-q_r(p))}{2\,q_r(p)(q_r^2+q_i^2)}e^{-i\,q_r\,z}\,\theta(-z)\\ +i\,&\frac{\mathbf{L}(p,i\,q_i(p))}{2\,q_i(p)(q_r^2+q_i^2)}e^{-q_i\,z}\,\theta(z) + i\,\frac{\mathbf{L}(p,-i\,q_i(p))}{2\,q_i(p)(q_r^2+q_i^2)}e^{q_i\,z}\,\theta(-z)\Big]e^{i\,p\,x} - \frac{1}{\eta_0}\mathbf{I}_z\delta(\mathbf{r}),\end{aligned} \tag{10}$$

where $\mathbf{I}_z$ is a matrix with only one nonzero component $(\mathbf{I}_z)_{33} = 1$.

4 Born approximation and energy attenuation

In present work we will limit ourselves to first, Born, approximation. From (9) it follows that

$$\begin{aligned} r(p) &= -\frac{i}{|\boldsymbol{a}^{(1)}|^2}\left(\boldsymbol{a}^{(1)},\,\frac{\mathbf{L}(p,q_r(p))}{2\,q_r(p)(q_r^2+q_i^2)\eta_0}\int\frac{dp_1}{2\pi}\mathbf{R}(p_1,q_r(p-p_1)+q_r(p))\,C^{inc}(p-p_1)\boldsymbol{a}^{(2)}(p-p_1)\right)\\ &\equiv i\int\frac{dp_1}{2\pi}Tr^{21}(p,p_1)C^{inc}(p-p_1)\\ t(p) &= C^{inc}(p) - \frac{i}{|\boldsymbol{a}^{(2)}|^2}\left(\boldsymbol{a}^{(2)},\,\frac{\mathbf{L}(p,-q_r(p))}{2\,q_r(p)(q_r^2+q_i^2)\eta_0}\int\frac{dp_1}{2\pi}\mathbf{R}(p_1,q_r(p-p_1)-q_r(p))\,C^{inc}(p-p_1)\boldsymbol{a}^{(2)}(p-p_1)\right)\\ &\equiv C^{inc}(p) + i\int\frac{dp_1}{2\pi}Tr^{22}(p,p_1)C^{inc}(p-p_1).\end{aligned} \tag{11}$$

Now we can evaluate attenuation of energy flux in the incident wave due to scattering. Considering the incident wave packet as being strongly localized at $p \sim p_0$, i.e. assuming that $C(p)$ has a peak at p_0, we introduce a quantity describing energy attenuation due to scattering

$$\triangle S = \langle \int_{p \sim p_0} \frac{dp}{2\pi} \left(|t(p)|^2 - |C^{inc}(p)|^2 \right) v_2 \rangle, \tag{12}$$

where brackets stands for ensemble averaging. For simplicity let us assume that density fluctuations are statistically homogeneous in x direction, so that $\langle \delta n^*(p_1)), \delta n(p_2) \rangle = w(p_1)\delta(p_1 - p_2)$. Hence after some straightforward calculation we find

$$\triangle S \simeq -\frac{W(p_0)}{v_2(p_0)} S^{inc}, \tag{13}$$

where $W(p_0)$ can easily be obtained from (11)

$$W(p_1)\delta(p_1 - p_2) = \int \frac{dp}{2\pi} v_2(p)(\langle (Tr^{22})^*(p, p_2) Tr^{22}(p, p_1) \rangle + \\ \langle (Tr^{21})^*(p, p_2) Tr^{21}(p, p_1) \rangle)$$

and S^{inc} stands for incident energy flux.

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