Electromagnetic Coupling to Curved Thick Wires

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Abstract

The full-wave transmission – line theory of the authors for thick, straight, and cylindrical wires is extended to those which are curved. It is assumed that the center line of the cylinder is located in the y-z plane. Therefore in the geometrical description of this thick wire only curvature of the center line will occur. Torsion is absent. Analytical solutions for special plane wave and ring – source excitations are discussed.
Keywords: Transmission line, high frequencies, thick curved wires.

1. Introduction

It has turned out that classical transmission – line theory does no longer meet modern requirements for complex systems, and the need for a great accuracy has led to the demand for better models. Such models have to be capable to model very complex geometries, like finite lines, nonuniform line conductors with curvature and torsion including periodic structures. They have to include radiation losses and non-TEM coupling at higher frequencies. Also for coaxial cables or cable – bundles at high frequencies azimuthal current distributions will become of remarkable amplitude and have therefore also to be treated in an extended new transmission – line theory.
In the present paper a new model is described which is based on the exact system of the electric field integral equations for the current and potential pair, which can be cast in the form of telegrapher equations. Therefore, most of existing techniques to solve such equations can be applied. In general, the new theory is computationally more efficient than other full-wave methods in certain applied problems. Other advantages are the possibilities to derive a physical interpretation of the new line parameters and to establish a relation between thick wires and multiconductor lines.
The mixed potential integral equations for a thick curved wire are simplified and completed to a thick, straight wire above a perfectly conducting ground. In this case the usual telegrapher equations are supplemented by an additional (third) equation for the angle (azimuthal) component of the current. The inductance per unit length and the capacitance per unit length become matrices of (in general) infinite dimensions. Besides the "longitudinal" inductance matrix one also obtains an "azimuthal" one. The three coupled telegrapher for the longitudinal component of the current, the azimuthal component of the current and for the scalar potential can be reduced to the usual two equations. In these equations, however, the line matrices become modified again. Another interesting result deals with the proximity effect. Evaluating this effect it is shown that a thick wire formally can be described by multiconductor TL equations. A short comparison with result of Sommerfeld [1] is made.

2. Conductor Geometry

It is assumed that the cylindrical conductor in free space is only slightly bent and that its center line lies in the y-z plane. The vector $\vec{r}_0(l)$ points to that curve (see Fig. 1). The parameter l is the arc length along the center curve of the conductor and the unit vector $\vec{e}_l(l)$ denotes the tangential unit vector:

$$\vec{e}_l(l) = \frac{\partial \vec{r}_0(l)}{\partial l}; \qquad \left|\vec{e}_l(l)\right| = 1 \tag{1}$$

This vector together with the unit normal and binormal vectors $\vec{n}(l)$ and $\vec{b}(l)$, respectively, form the so-calling co-moving Frenel – basis [2]:

$$\vec{n}(l) = \frac{1}{K(l)}\frac{\partial \vec{e}_l(l)}{\partial l}; \qquad \vec{b}(l) = \vec{e}_l(l)\times\vec{n}(l) = \vec{e}_x(l); \qquad \left|\vec{n}(l)\right| = \left|\vec{b}(l)\right| = 1 \tag{2}$$

This work is dedicated to the memory of our long-time friend and colleague, Carl Baum. We will always keep him in our memory and remember him as an extraordinary scientist, composer and warm – hearted person.

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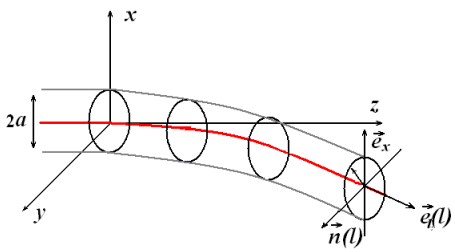

Fig. 1 Geometry of thick wire

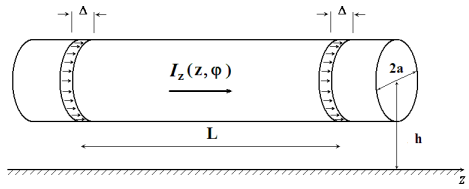

Fig. 2 Excitation of an infinite thick wire by two ring sources

Here $K(l)$ is the curvature of the curve $\vec{r}_0(l)$, a value, which is inverse to the radius of curvature $R(l)$ of the curve. Then one can introduce a local cylindrical coordinate system (ρ,φ,l) in the plane spanned by $\vec{n}(l)$ and $\vec{b}(l)$. Every point $\vec{r}(l,\rho,\varphi)$ inside the conductor volume V then can be addressed by these coordinates:

$$\vec{r}(l,\rho,\varphi)=\vec{r}_0(l)+\vec{n}(l)\rho\sin(\varphi)+\vec{e}_x\rho\cos(\varphi) \tag{3}$$

In particular, points on the surface of conductor are given by

$$\vec{r}(l,\varphi)=\vec{r}_0(l)+a\sin\varphi\vec{n}(l)+a\cos\varphi\vec{e}_x \quad \text{with } \rho=a \text{ (radius of the cylinder)} \tag{4}$$

In order to apply the boundary conditions for the electric field the corresponding Frenet – basis $(\tilde{\vec{e}}_l(l),\tilde{\vec{e}}_\rho(l),\tilde{\vec{e}}_\varphi(l))$ on the surface of the conductor has to be known. It is obtained by the successive differentiation of the vector $\vec{r}(l,\rho,\varphi)$ with respect to l, ρ, and φ resulting in

$$\tilde{\vec{e}}_\rho(l)=\vec{n}(l)\sin\varphi+\vec{e}_x\cos\varphi\,, \qquad \tilde{\vec{e}}_\varphi(l)=\vec{n}(l)\cos\varphi-\vec{e}_x\sin\varphi \tag{5a,b}$$

In this basis all volume (dV) and surface (dS) elements as well as operator representations have to be calculated. One simply achieves this using differential geometry [e.g.,2]. For the surface element dS one obtains

$$dS=a(1-K(l)a\sin\varphi)d\varphi dl\ , \tag{6}$$

for the gradient ∇ of a potential $\Phi(l,\varphi)$ and the divergence operator for a current $\vec{i}(l,\varphi)$

$$\nabla\cdot\Phi(l,\varphi)=\frac{1}{h_1}\frac{\partial\Phi}{\partial\rho}\tilde{\vec{e}}_\rho+\frac{1}{h_2}\frac{\partial\Phi}{\partial\rho}\tilde{\vec{e}}_\varphi+\frac{1}{h_3}\frac{\partial\Phi}{\partial l}\vec{e}_l(l)\,, \qquad \operatorname{div}\vec{i}=\frac{1}{h_1h_2h_3}\left(\frac{\partial}{\partial l}(h_1h_2i_l)+\frac{\partial}{\partial\varphi}(h_1h_3i_\varphi)\right) \tag{7 a,b}$$

Here h_1, h_2 and h_3 denote the Lame coefficients

$$h_1=1\,, \qquad h_2=\rho\,, \qquad h_3=1-K(l)\rho\sin\varphi \tag{8}$$

and the current vector $\vec{i}(l,\varphi)$ is represented on the surface of the wire by

$$\vec{i}(l,\varphi)=i_l(l,\varphi)\tilde{\vec{e}}_l+i_\varphi(l,\varphi)\tilde{\vec{e}}_\varphi \qquad (\rho=a\,,\quad i_\rho=0) \tag{9}$$

After these geometrical preliminaries one is prepared to formulate the physical equations.

3. Electromagnetic Equations

In this subsection the mixed potential integral equations (MPIE) shall be derived for the thick wire. Since the interest is focused on currents and potentials on the surface of the conductor one deals with surface – charges, - currents and – potentials. Using the continuity equation in frequency domain one derives the equation

$$q(l,\varphi)=-\frac{1}{j\omega}\frac{1}{1-Ka\sin\varphi}\left\{\frac{\partial i_l}{\partial l}+\frac{1}{a}\frac{\partial}{\partial\varphi}((1-Ka\sin\varphi)i_\varphi)\right\} \tag{10}$$

Insertion of this equation into the integral for the electrical potential Φ

$$\Phi(l,\varphi)=\frac{1}{4\pi\varepsilon_0}\int_S G(l,\varphi;l',\varphi')q(l',\varphi')dS' \tag{11}$$

yields the first MPIE – equation

$$\Phi(\varphi,l)+\frac{1}{j\omega4\pi\varepsilon_0}\int_{-\infty}^{\infty}dl'\int_0^{2\pi}ad\varphi'\left\{\frac{1}{a}\frac{\partial}{\partial\varphi'}(1-K(l')a\sin\varphi')\cdot i_\varphi(\varphi',l')+\frac{\partial}{\partial l'}i_l(\varphi',l')\right\}G(l,\varphi;l',k')=0 \tag{12}$$

with the Green's function of the thick wire in free space $G(l,\varphi;l',\varphi')=\exp\left(-jk\left|\vec{r}(\varphi,l)-\vec{r}(\varphi',l')\right|\right)/\left|\vec{r}(\varphi,l)-\vec{r}(\varphi',l')\right|$ (13)

The other two MPIE equations are obtained from the boundary conditions for the total electrical field:

$$\vec{E}_\tau^{sc}(l,\varphi)+\vec{E}_\tau^{in}(l,\varphi)=0 \tag{14}$$

The incident field $\vec{E}_\tau^{in}(l,\rho,\varphi)$ is known, whereas the scattered field is expressed with the aid of the electrical and magnetic potential:

$$\vec{E}_\tau^{sc}(l,\varphi)=-(\nabla\Phi)_\tau-j\omega\vec{A}_\tau \tag{15}$$

From eq. (7a) one concludes

$$(\nabla\Phi)_\tau=\frac{1}{h_3}\frac{\partial\Phi}{\partial l}\tilde{\vec{e}}_l+\frac{1}{h_2}\frac{\partial\Phi}{\partial\varphi}\tilde{\vec{e}}_\varphi \tag{16}$$

Equally one obtains from $\vec{A}$

$$\vec{A}=\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dl'\int_0^{2\pi}ad\varphi'(1-K(l')a\sin\varphi')\{i_l(l',\varphi')\tilde{\vec{e}}_l+i_\varphi(l',\varphi')\tilde{\vec{e}}_\varphi\}G(l,\varphi;l',k') \text{ with } A_l=\vec{A}(l,\varphi)\cdot\tilde{\vec{e}}_l(l,\varphi),\ A_\varphi=\vec{A}(l,\varphi)\cdot\tilde{\vec{e}}_\varphi(l,\varphi) \tag{17}$$

and eq. (14) the other two MPIE equations

$$\frac{1}{1-K(l)a\sin\varphi}\frac{\partial\Phi}{\partial l}+j\omega\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dl'\int_0^{2\pi}ad\varphi'(1-K(l')a\sin\varphi')\{(\vec{n}(l')-\vec{n}(l))\cdot\vec{e}_l(l)\cos\varphi' i_\varphi(\varphi',l')+\vec{e}_l(l)\cdot\vec{e}_l(l')i_l(\varphi_1,l_1)\}G(l,\varphi;l',\varphi')=E_l^{in}(l,\varphi) \tag{18}$$

$$\frac{1}{a}\frac{\partial\Phi}{\partial\varphi}+j\omega\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty}dl'\int_0^{2\pi}ad\varphi'(1-K(l')a\sin\varphi')\{[(\vec{n}(l)\cdot\vec{n}(l')-1)\cos\varphi\cos\varphi'+\cos(\varphi-\varphi')]i_\varphi(\varphi',l')+$$
$$+(\vec{n}(l)-\vec{n}(l'))\cdot\vec{e}_l(l')\cos\varphi i_l(\varphi',l')\}\cdot G(l,\varphi;l',\varphi')=E_\varphi^{in}(l,\varphi) \tag{19}$$

Here E_φ^{in} and E_l^{in} are φ and l components of the exciting external field on the boundary of the wire, respectively. Note that unlike the MPIE for the thin-wire case, the MPIE system contains three equations. The additional equation is necessary for the description of the azimuthal current components which does not occur in the thin – wire approximation. Note also that so far only one wire in free space is described which does not constitute a transmission line. The reference conductor is missing. The inclusion of reference conductor, (e.g., conducting ground plane) will just change the Green's functions in the above equations. In the next section a solution of MPIE are given for an infinite straight wire parallel to the perfectly conducting ground.

4. Full – Wave Transmission - Line Solution for a Thick Straight Wire

In the case of a straight wire above ground the MPIE eqs. (12), (18), and (19) have to be modified. First, the curvature $K(l)$ vanishes and second the co-moving basis simplifies to

$$\tilde{\vec{e}}_l(l,\varphi)=\vec{e}_z(z);\quad (l=z);\quad \vec{n}(l)=\vec{e}_y(z);\qquad \text{and}\qquad \tilde{\vec{e}}_\varphi(l,\varphi)=\vec{e}_\varphi(\varphi)=(-\sin\varphi,\cos\varphi,0) \tag{20}$$

Also, the Green's functions change, due to the reflection of the perfectly conducting ground. The Green's functions occurring in eqs. (12) and (18) have to be replaced by the function [3]

$$g_z(\varphi,\varphi_1,z)=g(\varphi,\varphi_1,z)-g_{ref}(\varphi,\varphi_1,z);\qquad g(\varphi,\varphi_1,z)=\frac{\exp\left(-jk\sqrt{4a^2\sin^2((\varphi-\varphi_1)/2)+z^2}\right)}{\sqrt{4a^2\sin^2((\varphi-\varphi_1)/2)+z^2}} \tag{21 a,b}$$

and

$$g_{ref}(\varphi,\varphi_1,z)=\frac{\exp\left(-jk\sqrt{\tilde{\rho}^2(\varphi,\varphi_1)+z^2}\right)}{\sqrt{\tilde{\rho}^2(\varphi,\varphi_1)+z^2}};\qquad \tilde{\rho}^2(\varphi,\varphi_1)=4h^2+4ah(\cos\varphi+\cos\varphi_1)+4a^2\cos^2\left(\frac{\varphi+\varphi_1}{2}\right) \tag{22 a,b}$$

The Green's function in (3.11) becomes $\quad g_\varphi(\varphi,\varphi_1,z)=g(\varphi,\varphi_1,z)-g_{ref}(\varphi,\pi-\varphi_1,z)$ (23)

Now it is the task to solve these modified equations. In Ref. [3] the solution was obtained by taking advantage of the translation and the (approximately) rotation - symmetry of the conductor configuration. Thus all important quantities could be represented as Fourier transforms in terms of the complete set of modal functions. A special excitation turned out to be of special interest: The excitation by ring sources. The infinite thick wire was excited by two ring sources spaced by the distance L (see Fig. 2) and the induced total current $I_z(z)$ and the averaged potential $\Phi(z)$ were described by the full – wave transmission line theory [3, 4]. First, it is assumed that the sources are independent. Second the assumption was made that the action on the system of the first source with unit amplitude results in a current $Y_1(z)$ and in the potential $K_1(z)$ along the line. The action of the second source with unit amplitude led to the current $Y_2(z)$ and potential $K_2(z)$ along the line. One has

$$Y_1(z) = I_{z,\delta}(z)/U_0\,; \qquad K_1(z) = \Phi_\delta(z)/U_0\,; \qquad Y_2(z) = I_{z,\delta}(z-L)/U_0\,; \qquad K_2(z) = \Phi_\delta(z-L)/U_0\,; \qquad (24\text{ a,b,c,d})$$

with (see Ref. [3])

$$\Phi_\delta(z) = \frac{U_0}{2}\big(h(z) - h(-z)\big)\, e^{-jk|z|} \qquad I_{z,\delta}(z) = \frac{2jkU_0}{\eta_0}\int_{-\infty}^{\infty} \frac{dk_1 e^{-jk_1 z}}{k_1^2 - k^2 + j\delta}\Big(\big[G_z(k,k_1)\big]^{-1}\Big)_{0,0} \qquad (24\text{ a,b})$$

where $\big[G_z\big]_{m,m_1} = -j\pi\Big\{H_m^{(2)}(\tilde{k}a)J_m(\tilde{k}a)\delta_{m,m_1} - (-1)^{m+m_1} J_m(\tilde{k}a)J_{m_1}(\tilde{k}a)H_{m+m_1}^{(2)}(2\tilde{k}h)\Big\}$ for $\tilde{k}^2 = k^2 - k_1^2 > 0$ (25)

It can be proved [3] that the quantities $\Phi(z)$ and $I_z(z)$ fulfill the following differential equations with parameter matrix $[P(z)]$:

$$\frac{d}{dl}\begin{pmatrix}\Phi(z)\\ I_z(z)\end{pmatrix} + j\omega\begin{bmatrix}P_{11}(z) & P_{12}(z)\\ P_{21}(z) & P_{22}(z)\end{bmatrix}\cdot\begin{pmatrix}\Phi(z)\\ I_z(z)\end{pmatrix} = 0\,, \qquad \big[P(z)\big] = -\frac{1}{j\omega}\left(\frac{d}{dz}\begin{bmatrix}K_1(z) & K_2(z)\\ Y_1(z) & Y_2(z)\end{bmatrix}\right)\begin{bmatrix}K_1(z) & K_2(z)\\ Y_1(z) & Y_2(z)\end{bmatrix}^{-1} \qquad (26)$$

The parameters are complex-valued, length-dependent, and depend on the geometry of the system. The non-classical part of the parameters (imaginary parts of the non-diagonal elements and diagonal elements) are connected with radiation. Using the TEM approximation it can be shown that for low frequencies *(kh<<1)* the parameter matrix approaches the one of classical TL theory. The parameter P_{12} becomes the classical inductance per-unit length and the parameter P_{21} becomes the classical capacitance per-unit length.

As an example consider an infinite line excited by two lumped sources. The height of the line is *h=0.5 m*, the radius of the wire is *a=0.25 m*, the distance between the sources is *L=10 m*, the frequency of excitation is *f=0.238 GHz* (*k=5* m^{-1}, *kh=2.5>~1*). Inductance and capacitance-like elements of the parameter matrix *[P(z)]* calculated from eq. (26) will be shown. For comparison also the FWTL parameters for a thin wire *a=1 cm* calculated by the approach given in [5] will be presented.

As for the thin-wire case, the parameters have singularities near the terminal region and smooth z- dependencies limitations to their classical values in the central part of the line. It is interesting to note, that a change of the radius of the wire influences the real part of the parameters (which is connected with the stored electrical and magnetic energy in the neighborhood of the wire) dramatically, but practically does not influence the imaginary part of the parameters (which is connected with radiation [4], [5]).

Another interesting case of excitation is the excitation by an incident vertically – polarized plane wave [6]. Then all Fourier components of the φ - component of the current i_φ vanish and than also i_φ itself. The i_z components remain preserved and keep their φ - dependency. A numerical example of azimuthal dependency of the axial current will be given. This example demonstrates the possibility to describe the proximity effect by the modal method. Also the analytical TEM solution of Sommerfeld [1] is compared with the result of the used modal TEM approach. With a sufficient number of (Fourier -) terms one obtains a very good agreement for the results.

5. Conclusion

The MPIE have be been derived for a curved cylindrical wire in free space which axis was described by a curve placed in the y-z plane (no torsion). After simplifying these equations for a thick, straight wire parallel to a conducting ground they were solved for special excitations by application of representations of the translation and rotation groups in terms of Bessel functions as well as by full-wave transmission – line equations. The proximity effect was demonstrated.

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7. References

1. Sommerfeld A., *Electrodynamics,* Academic Press, NY,1952.
2. B. A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern Geometry - Methods and Applications*, Springer 1991.
3. Nitsch, J., Tkachenko, S., "Full-Wave Transmission Line Theory for Thick Cylindrical Wires", *ISTET 2009*, Lübeck, Germany.
4. Nitsch J., Tkachenko S., Global and Modal Parameters In the Generalized Transmission Line Theory and Their Physical Meaning, *Rad. Sci. Bull.*, 312, March 2005, 21-31.
5. Nitsch, J., Tkachenko, S., "Physical Interpretation of the Parameters in the Full-Wave Transmission Line Theory", *ISTET 2009,* Lübeck, Germany
6. Nitsch J., Tkachenko S., "Newest Developments in Transmission-Line Theory and Applications", *Ultra-Wideband, Short-Pulse Electromagnetics* 7, Springer 2007, 21-22.