Sampling Distributions for Undermoded Mode-Stirred Reverberation Chambers

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Abstract

We derive the sampling probability density function (pdf) of a nonlocal random electromagnetic field, field amplitude and intensity in an undermoded chamber, i.e., in a statistically inhomogeneous time-varying environment generated by a combined spatio-temporal stochastic process. The inhomogeneous field is represented as a subset (sample) of a homogeneous field (ensemble). The sample statistics of the inhomogeneous field are governed by the number of spatial degrees of freedom, in addition to the number of temporal (stir) degrees of freedom.

1 Introduction

In recent work [1], we derived sampling pdfs for random electromagnetic (EM) fields that are local in space and/or spatially homogeneous in a statistical sense, i.e., whose statistical properties are independent of location of the point of observation. In particular, the ensemble mean value and standard deviation are independent of spatial location in this idealized scenario. When calculated for realistic sample sets, this sample standard deviation exhibits fluctuations caused by the finite size of a sample set of field data (realization). These fluctuations vary randomly from sample set to sample set, causing the (sample) standard deviation to become itself a random variable, in addition to the randomness of the field itself. The particular value of the sample standard deviation calculated from a particular sample data set is then a realized value of the sampling standard deviation. The smaller the sample set, the larger the fluctuations around its expectation value, i.e., the ensemble standard deviation.

On the other hand, in earlier work, we investigated the effect of *deterministic* inhomogeneity on the field pdf [2, 3] and on its spatial correlation properties [4]. There, the inhomogeneity was induced by an impedance boundary (dielectric, magnetic or conducting half-space), as a canonical configuration and simplest departure from an unbounded homogeneous propagation environment for the random field. While extension of the methodology and results to multilayered stratified media is straightforward, more complicated types of boundary-value problems are more problematic to solve, at least analytically. Some limited results on the average value and standard deviation for 2-D and 3-D semi-infinite corners have been derived, and indeed the methodology of [2, 3] could be applied to them to find the pdf.

In this paper, we investigate another canonical configuration, viz., a *statistically* inhomogeneous field. While this could model the pdf of a multiple-scattered field in either a heterogeneous or turbulent *medium*, we shall not perform a boundary-value calculation. Instead, we focus on the resulting statistical inhomogeneity of the field only and make the ansatz that the statistical parameters of the field thus acquire (spatial or temporal) randomness to reflect the statistical heterogeneity. It is now clear that the ensemble pdf of statistically inhomogeneous fields can be related to the sampling pdf of a statistically homogeneous field. Indeed, in the case of Gaussian statistical inhomogeneity, these pdfs have the same functional form, because the (Gauss normal distributed) mathematical fluctuations of the standard deviation in sampled finite sets for a sampled homogeneous fields can then be re-interpreted as physical fluctuations for the ensemble inhomogeneous field. Therefore, in this paper we considering the *sampling* pdf of the statistically *inhomogeneous* field (1-D stochastic process). It follows that the results of such an analysis also represent the *ensemble* pdf for a doubly-stochastic inhomogeneous field, e.g., for the ensemble EM field generated by a 2-D space-time stochastic process. Such a situation occurs, for example, in a MT/MSRC (*a fortiori* in its so-called "undermoded" regime, where the mode density is relatively small) where spatial and temporal fluctuations of the field, across the cavity and in the course of the stirring process, respectively, occur independently (regardless whether or not the mean value, standard deviation, etc. of these fluctuations are equivalent, on the basis of the principle of ergodicity). More generally, mesoscopic dynamical systems exhibit statistical fluctuations whose ensemble distribution

Physically, the view purported in this paper is that probability distributions of an undermoded system may be characterized by a finite-size *sampling distribution* for samples of finite size N, associated with the asymptotic distribution of the ensemble corresponding to the ideal overmoded system (infinite number of degrees of freedom, $N \to +\infty$). This is made plausible by the fact that for increasing physical size of the cavity the field the ratio of the field correlation length to the characteristic dimension of the cavity approaches zero and the field becomes approaches statistical homogeneity.

Time-varying and spatial-statistically homogeneous random fields are characterized by pdfs whose distribution parameters (e.g., average and standard deviation) have estimated values that show sampling fluctuations, as a result of the finite value of ν governing the local field. These variations give rise to bivariate fluctuations and larger uncertainty (wider confidence intervals) of the sampled field, compared to the corresponding ensemble distribution.

The physical origin of the limitation of the number of degrees of freedom can be twofold. First, the potential (i.e., maximum attainable) number of *statistically independent* realizations, $N_{\rm max}$, may be restricted, even if an unlimited number of different states of the system were generated. In a MT/MSRC, this case represents the undermoded regime, while in a relatively small sample of a random medium it refers to the case of a relatively small loading fraction of inclusions. In this case, even the ensemble df does not possess Gauss normal statistical properties. Secondly, $N_{\rm max}$ may be unlimited but, for economical or other reasons, the sample size may have to be severely restricted ($N \ll N_{\rm max}$). Thus, in this case, while the number of degrees of freedom is potentially large, the generated number of degrees of freedom is relatively small.

2 Electric or magnetic field

Consider a spatially local analytic random electric field $E(\underline{r}) = E'(\underline{r}) - {\rm j} E''(\underline{r})$ as received by an intentional or unintentional sensor (antenna) or probe at location $\underline{r}$ in a time-varying multiple-scattering environment. A harmonic time dependence $\exp({\rm j}\omega t)$ is assumed and suppressed. If this field is made up of an arbitrarily large (theoretically infinite) number of contributions (independently fluctuating partial fields) forming a random walk, then on account of the CLT (valid under very general but definite conditions [5]) we can assert that the associated likelihood of $E'^{(\prime)}$ is given by

$$f_{E'^{(\prime)}|S_{E'^{(\prime)}}}(e'^{(\prime)}|s_{E'^{(\prime)}}) = \frac{\exp\left[-\frac{\left(e'^{(\prime)} - m_{E'^{(\prime)}}\right)^2}{2\, s^2_{E'^{(\prime)}}}\right]}{\sqrt{2\pi}\; s_{E'^{(\prime)}}} \tag{1}$$

In [1], we demonstrated that the sampling pdf of $E'^{(\prime)}(\underline{r})$ of a statistically homogeneous field is given by

$$f_{E'^{(\prime)}}(e'^{(\prime)}; N) \quad = \quad \frac{C_{E'^{(\prime)}}}{\sigma_{E'^{(\prime)}}} \left(\frac{|e'^{(\prime)}|}{\sigma_{E'^{(\prime)}}}\right)^{\frac{pN}{2}-1} K_{\frac{pN}{2}-1}\left(\sqrt{pN-1}\,\frac{|e'^{(\prime)}|}{\sigma_{E'^{(\prime)}}}\right) \tag{2}$$

when the random field is governed by N degrees of freedom, with

$$C_{E'^{(\prime)}} \triangleq \frac{(pN-1)^{\frac{pN}{4}}}{2^{\frac{pN}{2}}\sqrt{\pi}\,\Gamma\left(\frac{pN-1}{2}\right)} \tag{3}$$

This sampling pdf can now be interpreted as the marginal sampling pdf of the local field in the case of a statistically inhomogeneous field. The sampling pdf of its nonlocal field is then obtained by considering $\sigma_{E'^{(\prime)}}$ and N in (2) to be a local parameter values at $\underline{r}$, which are themselves sample values $s_{\mathcal{E}'^{(\prime)}}$ $N_{\rm t}$ for the second stochastic process with its own associated ensemble standard deviation $\sigma_{\mathcal{E}'^{(\prime)}}$. In other words,

$$\sigma_{E'^{(\prime)}} \to s_{\mathcal{E}'^{(\prime)}}(\underline{r}), \quad N \to N_{\rm t}(\underline{r}) \tag{4}$$

whence

$$f_{\mathcal{E}'^{(\prime)}|S_{\mathcal{E}'^{(\prime)}}}(\epsilon'^{(\prime)}|s_{\mathcal{E}'^{(\prime)}}; N_{\rm t}) = \frac{C_{\mathcal{E}'^{(\prime)}|S_{\mathcal{E}'^{(\prime)}}}}{s_{\mathcal{E}'^{(\prime)}}} \left(\frac{|\epsilon'^{(\prime)}|}{s_{\mathcal{E}'^{(\prime)}}}\right)^{\frac{pN_{\rm t}}{2}-1} K_{\frac{pN_{\rm t}}{2}-1}\left(\sqrt{pN_{\rm t}-1}\frac{|\epsilon'^{(\prime)}|}{s_{\mathcal{E}'^{(\prime)}}}\right) \tag{5}$$

in which $C_{\mathcal{E}'^{(\prime)}|S_{\mathcal{E}'^{(\prime)}}} \equiv C_{E'^{(\prime)}}$ and where $S_{\mathcal{E}'^{(\prime)}}$ as a variate function of position exhibits a spatial $\chi_{pN_{\rm s}-1}$ distribution with ensemble standard deviation $\sigma_{S_{\mathcal{E}'^{(\prime)}}}$, i.e.,

$$f_{S_{\mathcal{E}'^{(\prime)}}}(s_{\mathcal{E}'^{(\prime)}}; N_{\rm s}) = \frac{C_{S_{\mathcal{E}'^{(\prime)}}}}{\sigma_{S_{\mathcal{E}'^{(\prime)}}}} \left(\frac{s_{\mathcal{E}'^{(\prime)}}}{\sigma_{S_{\mathcal{E}'^{(\prime)}}}}\right)^{pN_{\rm s}-2} \exp\left[-\mathcal{M}_{\rm s}\left(\frac{s_{\mathcal{E}'^{(\prime)}}}{\sigma_{S_{\mathcal{E}'^{(\prime)}}}}\right)^2\right] \tag{6}$$

with

$$C_{S_{\mathcal{E}'^{(\prime)}}} \triangleq \frac{2\,\mathcal{M}_{\rm s}^{\frac{pN_{\rm s}-1}{2}}}{\Gamma\left(\frac{pN_{\rm s}-1}{2}\right)}, \qquad \mathcal{M}_{\rm s} \triangleq \frac{pN_{\rm s}-1}{2} - \left(\frac{\Gamma\left(\frac{pN_{\rm s}}{2}\right)}{\Gamma\left(\frac{pN_{\rm s}-1}{2}\right)}\right)^2 \tag{7}$$

Thus, the sampling pdf of the nonlocal analytic field follows finally as

$$\begin{aligned} f_{\mathcal{E}'^{(\prime)}}(\epsilon'^{(\prime)}; N_{\rm t}, N_{\rm s}) &= \int_0^{+\infty} f_{\mathcal{E}'^{(\prime)}|S_{\mathcal{E}'^{(\prime)}}}(\epsilon'^{(\prime)}|s_{\mathcal{E}'^{(\prime)}}; N_{\rm t}) f_{S_{\mathcal{E}'^{(\prime)}}}(s_{\mathcal{E}'^{(\prime)}}; N_{\rm s}) {\rm d}s_{\mathcal{E}'^{(\prime)}} \\ &= \frac{C_{\mathcal{E}'^{(\prime)}|S_{\mathcal{E}'^{(\prime)}}}\ C_{S_{\mathcal{E}'^{(\prime)}}}}{\sigma^{pN_{\rm s}-1}_{S_{\mathcal{E}'^{(\prime)}}}} |\epsilon'^{(\prime)}|^{\frac{pN_{\rm t}}{2}-1} \int_0^{+\infty} s_{\mathcal{E}'^{(\prime)}}^{-\frac{pN_{\rm t}}{2}+pN_{\rm s}-2} \\ &\quad \times \exp\left[-\mathcal{M}_{\rm s}\left(\frac{s_{\mathcal{E}'^{(\prime)}}}{\sigma_{S_{\mathcal{E}'^{(\prime)}}}}\right)^2\right] K_{\frac{pN_{\rm t}}{2}-1}\left(\sqrt{pN_{\rm t}-1}\frac{|\epsilon'^{(\prime)}|}{s_{\mathcal{E}'^{(\prime)}}}\right) {\rm d}s_{\mathcal{E}'^{(\prime)}} \end{aligned} \tag{8}$$

Here, we assume that $N_{\rm t}$ in (4) is independent of $\underline{r}$ and that $N_{\rm s}$ is independent of time, so that both can be considered as constants. By extension, if either or both carry their own fluctuations, (8) is itself a marginal pdf with respect to fluctuations of $N_{\rm t}(\underline{r})$ and/or $N_{\rm s}(t)$, whence further integrations with respect to prior pdfs $f_{N_{\rm t}}(n_{\rm t})$ and/or $f_{N_{\rm s}}(n_{\rm s})$ are necessary in order to arrive at the sampling field pdf in this case.

In the limit of a spatially homogeneous field ($N_{\rm s} \to +\infty$), the bivariate stochastic process reduces to a univariate one, i.e., we $f_{\mathcal{E}'^{(\prime)}}(\epsilon'^{(\prime)}; N_{\rm t}, N_{\rm s} \to +\infty) \equiv f_{E'^{(\prime)}}(e'^{(\prime)}; N_{\rm t})$ and is given by the expression in (5).

Figs. **??** and **??** and show the sampling pdf (12) of a Cartesian field component ($p = 1$), for an ergodic field ($N_{\rm t} = N_{\rm t} \triangleq N$) and for a nonergodic field ($N_{\rm t} = 5$), respectively. The effect of the additional statistical inhomogeneity is seen to result in the sampling pdf now exhibiting larger spread (lower maximum value and heavier right-hand tails) and, hence, increased statistical uncertainty and fluctuation levels.

3 Field intensity, energy density, power

The exposition of the calculation of the pdf of the field intensity $\mathcal{U} = |\mathcal{E}|^2$ – as well as the energy density or power, which are proportional to $\mathcal{U}$ – follows the same methods as in [1, Sec.III]. Thus, from the conditional and "prior" pdfs

$$f_{\mathcal{U}|S_{\mathcal{U}}}(\mu|s_{\mathcal{U}}; N_{\rm t}) = \frac{C_{\mathcal{U}|S_{\mathcal{U}}}}{s_{\mathcal{U}}} \left(\frac{\mu}{s_{\mathcal{U}}}\right)^{\frac{1}{2}\left[p(N_{\rm t}+1)-\frac{5}{2}\right]} K_{p(N_{\rm t}-1)-\frac{1}{2}}\left(2\sqrt{\sqrt{p}\left(pN_{\rm t}-\frac{1}{2}\right)}\sqrt{\frac{\mu}{s_{\mathcal{U}}}}\right) \tag{9}$$

$$f_{S_{\mathcal{U}}}(s_{\mathcal{U}};N_{\rm s}) = \frac{C_{S_{\mathcal{U}}}}{\sigma_{S_{\mathcal{U}}}}\left(\frac{s_{\mathcal{U}}}{\sigma_{S_{\mathcal{U}}}}\right)^{pN_{\rm s}-\frac{3}{2}} \exp\left(-\sqrt{pN_{\rm s}-\frac{1}{2}}\,\frac{s_{\mathcal{U}}}{\sigma_{S_{\mathcal{U}}}}\right) \tag{10}$$

with

$$C_{\mathcal{U}|S_{\mathcal{U}}} \triangleq \frac{2}{\Gamma(p)\Gamma\left(pN_{\rm t}-\frac{1}{2}\right)} p^{\frac{1}{4}\left[p(N_{\rm t}+1)-\frac{1}{2}\right]}\left(pN_{\rm t}-\frac{1}{2}\right)^{\frac{1}{2}\left[p(N_{\rm t}+1)-\frac{1}{2}\right]}, \qquad C_{S_{\mathcal{U}}} \triangleq \frac{\left(pN_{\rm s}-\frac{1}{2}\right)^{\frac{1}{2}\left(pN_{\rm s}-\frac{1}{2}\right)}}{\Gamma\left(pN_{\rm s}-\frac{1}{2}\right)} \tag{11}$$

it follows that

$$\begin{aligned} f_{\mathcal{U}}(\mu;N_{\rm t},N_{\rm s}) &= \int_0^{+\infty} f_{\mathcal{U}|S_{\mathcal{U}}}(\mu|s_{\mathcal{U}};N_{\rm t}) f_{S_{\mathcal{U}}}(s_{\mathcal{U}};N_{\rm s}) {\rm d}s_{\mathcal{U}} \\ &= \frac{C_{\mathcal{U}|S_{\mathcal{U}}}\, C_{S_{\mathcal{U}}}}{\sigma_{S_{\mathcal{U}}}^{pN_{\rm s}-\frac{1}{2}}} \mu^{\frac{1}{2}\left[p(N_{\rm t}+1)-\frac{5}{2}\right]} \int_0^{+\infty} s_{\mathcal{U}}^{-\frac{pN_{\rm t}}{2}+pN_{\rm s}-\frac{p}{2}-\frac{5}{4}} \\ &\quad \times \exp\left(-\sqrt{pN_{\rm s}-\frac{1}{2}}\,\frac{s_{\mathcal{U}}}{\sigma_{S_{\mathcal{U}}}}\right) K_{p(N_{\rm t}-1)-\frac{1}{2}}\left(2\sqrt{\sqrt{p}\left(pN_{\rm t}-\frac{1}{2}\right)}\sqrt{\frac{\mu}{s_{\mathcal{U}}}}\right) {\rm d}s_{\mathcal{U}} \end{aligned} \tag{12}$$

4 Conclusion

In this paper, we derived the sampling probability density function for a statistically inhomogeneous analytic field and associated energy density, as found in an undermoded reverberation chamber. The results constitute an extension of previous work on sampling distributions of overmoded fields on one hand, and ensemble distributions for deterministically inhomogeneous fields as found e.g. near a PEC or dielectric halfspace. The results are useful in determining the intrinsic field uncertainty for undermoded fields with greater accuracy.

5 References

References

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