Physical Considerations on Complex Cavities in Undermoded Regime

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Abstract

In this paper, a theoretical investigation about undermoded mode-stirred reverberation chambers is presented and discussed. This is based on the assumption of a *sinc*-correlated vector plane-wave angular spectrum. We extend the plane wave superposition to include such correlated scenario and find an integral representation for the total field coherency (single point correlation). This clearly exhibits spatial inhomogeneity and agrees with both Sommerfeld's far-field radiation condition and Berry's conjecture for chaotic scattering in confined wave systems. Finally, we find a closed-form solution for the coherency at the center of the reference frame, serving as a normalization factor to compare with measurements in complex inhomogeneous environments. Several physical considerations arise from this novel analytic representation.

1 Introduction

In reverberation chambers and chaotic billiards, the hypothesis of an idealized delta-correlated Gaussian complex Cartesian partial field (particle wavefunction) is valid in the semiclassical limit, and asymptotic statistics of the overall field/energy can be easily derived [1]. In the presence of a small ndof or a perturbation creating a few statistically independent configurations, the Gaussian idealization does no longer hold, and the underlying statistical assumptions require the inclusion of a partially coherent field. Such a hypothesis is physically consistent with the presence of statistical inhomogeneity due to a reduced number of modes building the local ensemble field. In this paper, a *first principle* model for the undermoded regime is presented and discussed. Starting from consistent second-order statistical assumptions on partial fields, the auto-coherency of the total field is calculated. As addressed in [1], a focus on the cross-spectral coherency of the field leads to an average energy density expression, which is crucial in the calculation of the received power inside the chamber.

The present investigation serves to elucidate the still unclear properties of a statistically inhomogeneous random field, arising inside an undermoded RC, i.e., when the chamber is operated at frequencies $f < LUF$, its lowest usable frequency [2, 3]. Whilst the occurrence of partial polarization and anisotropy have been investigated and analytical formulations developed with the aid of the Wiener coherency matrix [4], field heterogeneity structure caused by pure undermoded regime awaits the formulation of a rigorous model on a physical basis.

Assuming a *sinc* correlated partial field [5, 6], we derive an integral expression of the average power density (intensity). Such an expression is purely real, according to the Sommerfeld's radiation conditions, and depends on the *cosine* of the electrical length, according to the Berry's conjecture for a chaotic cavity. It is also proved that the joint existence of inhomogeneity and anisotropy is indeed possible.

The problem with undermoded stirred chambers is not limited to the ensemble depth, but also to the number of interfering partial fields components (few modes/rays) and how they are correlated within the cavity bandwidth and over the time/frequency stirring process.

The fundamental point towards the extension of the plane-wave spectrum (pws) to undermoded scenarios is the definition of a generalized 2D correlation function for random fields.

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2 Statistical assumptions for undermoded regime

In first instance, it shall be stated that *two partial fields are correlated if and only if they become statistically dependent and interfere over a finite angular (solid) sector.* Such a property leads to a simple geometrical interpretation: the propagation of two plane waves is correlated when their wave-vector directions lie across a spherical cap of angular sector $\Delta\tilde{\Omega} = (\theta_1 - \theta_2)(\phi_1 - \phi_2)$. Although this gives an easy way of visualizing the physical behavior of random fields, the functional form of the correlation function still needs a rigorous definition. In so doing, further reasonable assumptions are required. In the frame of two-dimensional random fields, the local averaging process is defined over a rectangular area A in the plane $\widehat{\theta\phi}$ [5]. Obviously, such a representation is allowed by the orthogonality of $\hat{\theta}$ and $\hat{\phi}$ in the spherical coordinates reference system. Therefore, the resulting correlation area is $A = \Delta\tilde{\Omega} = \Omega_2 - \Omega_1$. A partial random field is δ-correlated, exhibiting functional dependence on $\delta(\Omega_2 - \Omega_1)$. The notation employed here does not mean areas in the (θ, ϕ)-plane, but distances between points among A. Only after integration that is necessary to collect the plane-wave superposition, they become areas [5, Eq. (6.1.1)]. Then, orthogonality plus (induced) statistical independence lead to a separable structure of correlation, spectral density and variance structures, namely [5]

$$\rho(\Omega_1, \Omega_2) = \rho(\theta_1, \theta_2; \phi_1, \phi_2) = \rho_\theta(\theta_1, \theta_2)\, \rho_\phi(\phi_1, \phi_2) = \rho_\theta\left(\Delta\tilde{\theta}\right) \rho_\phi\left(\Delta\tilde{\phi}\right) \ , \tag{1}$$

where the functional form of ρ_θ and ρ_ϕ must be physically sustainable w.r.t. limiting cases. First of all, in the (θ, ϕ)-plane, the correlation must be double periodic, it shall reduce to a δ function by taking the limit for vanishing correlation length, while it shall reduce to a constant (unity) by considering completely correlated, e.g., single mode environments (infinite correlation length).

All these conceptual requirements seem to point toward the non-normalized $\mathrm{sinc}(x) = \sin(x)/x$ as a proper trigonometric functional form to generalize random field correlation. Thus, for the complex Cartesian components it follows that [5]

$$\rho_\theta\left(\Delta\tilde{\theta}\right) = l_\theta \frac{\mathrm{sinc}\left[\left(\frac{\theta_1 - \theta_2}{l_\theta}\right)\right]}{(\theta_1 - \theta_2)} \ , \tag{2}$$

$$\rho_\phi\left(\Delta\tilde{\phi}\right) = l_\phi \frac{\mathrm{sinc}\left[\left(\frac{\phi_1 - \phi_2}{l_\phi}\right)\right]}{(\phi_1 - \phi_2)} \ , \tag{3}$$

where l_θ and l_ϕ are the correlation lengths of the 3D random field in elevation and azimuthal directions, respectively. Expressions like (3) have been widely used also in optical theory, mainly in investigation of statistical properties of a coherent beam crossing apertures [7]. Within this frame, l_θ and l_ϕ can be expressed in terms of the aperture geometry. Therefore, the partial field (angular) cross-spectral coherency takes the usual form [1], now with the generalized functional expression

$$\langle \mathbf{F}(\Omega_1)\, \mathbf{F}^*(\Omega_2) \rangle = 4C\, \rho_\theta\left(\Delta\tilde{\theta}\right) \rho_\phi\left(\Delta\tilde{\phi}\right) \ , \tag{4}$$

where C is the angular correlation coefficient of the associated ensemble field [1]. Assuming such a partial field coherence, it turns out that, when the correlation length increases, the inhomogeneous angular sector widens and more correlation cells become coupled, thus exchanging energy through the fluctuation governed by the stirring process. Now, it is easy to retrieve canonical limits of a purely random (uncorrelated) field by taking the limit

$$\lim_{l_{\theta,\phi} \to 0} \frac{1}{l_{\theta,\phi}} \mathrm{sinc}\left[\frac{x}{l_{\theta,\phi}}\right] = \delta(x) \ , \tag{5}$$

which produces the idealized δ-correlated process

$$\langle \mathbf{F}(\Omega_1)\, \mathbf{F}^*(\Omega_2) \rangle = \delta(\theta_1 - \theta_2)\ \delta(\phi_1 - \phi_2) \tag{6}$$

$$= \delta(\theta_1 - \theta_2, \phi_1 - \phi_2) \tag{7}$$

$$= \delta(\Omega_1 - \Omega_2) \ , \tag{8}$$

where the product measure of the n-dimensional δ process has been used, and the limit

$$\lim_{l_{\theta,\phi}\to\infty} \frac{1}{l_{\theta,\phi}} \mathrm{sinc}\left[\frac{x}{l_{\theta,\phi}}\right] = 0 \ , \tag{9}$$

gives the deterministic scenario supported by the *mode orthogonality*.

3 Cross-spectral coherency and energy

Substitution of (4) into the pws [1, Eq. (11)] for two separate locations $\mathbf{r_1}$ and $\mathbf{r_2}$ leads to the total field cross-spectral coherency

$$\langle \boldsymbol{E}(\boldsymbol{r}_a)\boldsymbol{E}^*(\boldsymbol{r}_b)\rangle = 4C \iint_{\Omega_1} \iint_{\Omega_2} \rho_\theta\left(\Delta\tilde{\theta}\right)\rho_\phi\left(\Delta\tilde{\phi}\right) \mathrm{e}^{-\mathrm{j}(\boldsymbol{k}_1\cdot\boldsymbol{r}_a-\boldsymbol{k}_2\cdot\boldsymbol{r}_b)}\, \mathrm{d}\Omega_1\mathrm{d}\Omega_2 \ , \tag{10}$$

where the propagator argument does not vanish giving dependence on both location and direction of separation, i.e., inhomogeneity [8]. In general, the scalar product between the wavenumber $\boldsymbol{k}_i$ and a location $\boldsymbol{r}_j$ in a source-free region, reads

$$\boldsymbol{k}_i\cdot\boldsymbol{r}_j = -kr_j\left(\sin\theta_i\sin\theta_j\cos\phi_i\cos\phi_j + \sin\theta_i\sin\theta_j\sin\phi_i\sin\phi_j + \cos\theta_i\cos\theta_j\right) \ , \tag{11}$$

where $i = \{1, 2\}$ is related to the partial fields while $j = \{a, b\}$ is related to the direction of separation. Hence, the difference $\boldsymbol{k}_1\cdot\boldsymbol{r}_a - \boldsymbol{k}_2\cdot\boldsymbol{r}_b$ follows without analytical complications. Simplifications on further analysis can be obtained **by** considering the special case of *single-point correlation* [7], viz. $\boldsymbol{r}_a = \boldsymbol{r}_b = \boldsymbol{r}$, yielding the exponentiation of

$$\begin{aligned}(\boldsymbol{k}_1-\boldsymbol{k}_2)\cdot\boldsymbol{r} = -kr\left[\sin\theta\cos\phi\left(\sin\theta_1\cos\phi_1 - \sin\theta_2\cos\phi_2\right) + \right.\\ \left.\sin\theta\sin\phi\left(\sin\theta_1\sin\phi_1-\sin\theta_2\sin\phi_2\right) + \cos\theta\left(\cos\theta_1-\cos\theta_2\right)\right] \ ,\end{aligned} \tag{12}$$

where θ and ϕ define the direction of incidence. On substituting (12) into (10), this gives the squared magnitude of the total field $\left\langle|\boldsymbol{E}(\boldsymbol{r})|^2\right\rangle$, which is proportional to the random field energy.
Because of the linearity of the cavity field, the spatial evolution of cross-spectral coherency can be followed along separate Cartesian directions. More precisely, on substituting (2) and (3) into (10), the Cartesian dependencies of cross-spectral coherency for the total field in directions for $\hat{x}$, $\hat{y}$, $\hat{z}$ exhibit inhomogeneous nature. The assumption of double sinc correlation of the partial field leads to a *statistical inhomogeneity* through the propagators, because of the direct dependence of the total field cross-spectral coherence on the spatial location. An important matter is the universal influence of the cavity boundaries to the local undermoded cavity field. Even though solvable in a rigorous way, such a problem results in a very articulate mathematical formulation. Integral expression (10) exhibits *null imaginary parts*, according to the Sommerfeld far-field radiation conditions, and cosine-dependence on the electrical length kr, according to the Berry conjecture on semiclassical wavefunctions for chaotic cavities [9]

$$\begin{aligned}\left\langle|\boldsymbol{E}(x,y,z)|^2\right\rangle = 4C\int_0^\pi\int_0^{2\pi}\int_0^\pi\int_0^{2\pi} \frac{\sin\left(\frac{\theta_1-\theta_2}{l_\theta}\right)}{(\theta_1-\theta_2)}\frac{\sin\left(\frac{\phi_2-\phi_1}{l_\phi}\right)}{(\phi_2-\phi_1)}\\ \times\cos\left\{\left[kx\left(\sin\theta_1\cos\phi_1-\sin\theta_2\cos\phi_2\right)\right] + \left[ky\left(\sin\theta_1\sin\phi_1-\sin\theta_2\sin\phi_2\right)\right] + \right.\\ \left.\left[kz\left(\cos\theta_1-\cos\theta_2\right)\right]\right\}\ \sin(\theta_1)\sin(\theta_2)\,\mathrm{d}\phi_2\mathrm{d}\theta_2\mathrm{d}\phi_1\mathrm{d}\theta_1 \ .\end{aligned} \tag{13}$$

These implications are physically consistent as reactive power is not *present* in pws for local fields far away from boundaries [10]. Parenthetically, it is worth noticing that a closed-form solution is possible for the *origin's* single-point coherency

$$\begin{aligned}\left\langle|\boldsymbol{E}(0,0,0)|^2\right\rangle = 4C\int_0^\pi\int_0^{2\pi}\int_0^\pi\int_0^{2\pi} \frac{\sin\left(\frac{\theta_1-\theta_2}{l_\theta}\right)}{(\theta_1-\theta_2)}\frac{\sin\left(\frac{\phi_2-\phi_1}{l_\phi}\right)}{(\phi_2-\phi_1)}\ \sin(\theta_1)\sin(\theta_2)\,\mathrm{d}\phi_2\mathrm{d}\theta_2\mathrm{d}\phi_1\mathrm{d}\theta_1 = \\ 4C\left\{\left[-4l_\phi^2\sin^2\left(\frac{\pi}{l_\phi}\right) + 4l_\phi\,\mathrm{SinIntegral}\left(\frac{2\pi}{l_\phi}\right)\right] + \frac{F(l_\theta)}{l_\theta^2-1}\right\} \ ,\end{aligned} \tag{14}$$

where the singularity in $l_\theta = 1$ appears, and $F(l_\theta)$ is a very complicate function of l_θ. This is of particular interest in the comparison with measurements in arbitrary locations of inhomogeneous environment, where a normalization factor should be used: a convenient location could be taken as center of the cavity, the empirical evaluation of (14) could be used to derive the correlation length $l_\phi = l_\theta = l$, and the numerical evaluation of (13) would trace the spatial inhomogeneous power density flowing within the cavity. By following the presented procedure, a 3D Wiener coherency matrix could be defined upon further assumption and the Cartesian field cross-correlation, whence pdf is determined by canonical transformations [6].

4 Conclusion

We have presented an extension of pws to incorporate sinc-correlated partial random fields. No hypothesis on the Gaussianity of the underlying process has been made. An integral representation for the single point correlation has been found for the total field. This expression, clearly exhibiting spatial inhomogeneity, is physically representative of the power density flowing within the mode-stirred cavity. The integral expression agrees with both Sommerfeld's far-field radiation condition and with Berry's conjecture for chaotic scattering in confined wave systems. A closed-form solution for the coherency at the center of the reference frame has been derived. This constitutes a key normalization factor for comparison with measurements in complex inhomogeneous environments.

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