On the Diversity Performance of Compact Antenna Arrays

Michel T. Ivrlač and Josef A. Nossek

Institute for Circuit Theory and Signal Processing
Technische Universität München, D-80333 Munich, Germany
email: {ivrlac, nossek}@tum.de

***Abstract*—We report on our research on diversity performance of compact antenna arrays where the distance between neighboring antennas is (much) less than half a wavelength. In contrast to common belief, such compact arrays are able to deliver *excellent* diversity performance provided that a multiport network is connected between the array and the receiver which decouples the antenna ports. It turns out that the diversity performance does *not* change much as the antenna separation is reduced below half a wavelength – in fact, the diversity performance even *increases* somewhat with reduced antenna spacing. In an isotropic noise environment, and in the absence of heat loss, excellent diversity performance can be maintained even as the antenna separation is made arbitrarily small.**

I. Introduction

The electromagnetic waves which are emitted by a transmitter excite numerous scattered waves from obstacles which do not happen to be small compared to the wavelength. When there is relative mobility between the source and the obstacles, each scattered wave will have a slightly different frequency due to the Doppler effect. Their superposition therefore makes for a time-varying signal strength at a receive antenna, showing the characteristic occasional deep fades due to destructive superposition. A common attempt to lower the chance of such deep fades is to use *antenna diversity*, where the signals from several antennas are combined. If the antennas are spaced sufficiently far apart, chances are that the respective signal amplitudes will be (almost) uncorrelated and *simultaneous* deep fades become much less likely, thereby improving link reliability [1].

The amount of correlation between antenna signals essentially depends on two factors: 1) the angular distribution of the scattered waves, or more abstractly, on the angular power density, and 2) on the antenna separation [2]. In general, receiving substantial amount of power from a wide range of angles tends to make for low correlation, as does a large separation of antennas, preferably by several wavelengths [3]. Now in a compact antenna array, the distance between neighboring antennas is (much) smaller than half a wavelength. This close spatial proximity makes neighboring antennas perceive similar incident electromagnetic fields such that the correlation of antenna signals is expected to increase. Small antenna separation is, therefore, expected to lower the effectiveness of diversity combining, and ultimately, rendering diversity combining completely ineffective, as the antenna separation inside the array is reduced further towards zero and the antenna signals become coherent.

The standard argument given above ignores the electromagnetic interaction of closely spaced antennas. The electric current flowing in one antenna excites its own electromagnetic field which is felt and responded to by the other antennas in the array. This results in strong mutual antenna coupling in compact arrays. It turns out that this mutual antenna coupling has some, perhaps surprising, effects. For an array of two isotropic radiators, we show that with the help of a *lossless decoupling network, the array essentially retains all of its diversity performance as the antenna separation is reduced from half wavelength towards zero*. Such antenna arrays with decoupling network are realizations of *multimode* antennas [4].

II. Diversity Measure

We quantify diversity performance by means of the so-called *diversity measure* [5]. To this end, let

$$\boldsymbol{u} = \boldsymbol{h}s + \boldsymbol{n} \tag{1}$$

be a vector of N noisy »copies« of a signal s, received over different diversity branches (say from N different antennas). The components of the vector $\boldsymbol{n}$ are samples of complex, circularly symmetric, zero-mean, Gaussian noise with correlation matrix $\boldsymbol{R_n} = \mathrm{E}[\boldsymbol{n}\boldsymbol{n}^{\mathrm{H}}]$. Diversity combining of all N signals yields:

$$\hat{s} = \boldsymbol{w}^{\mathrm{H}}\boldsymbol{u}, \tag{2}$$

where the combining vector $\boldsymbol{w} \in \mathbb{C}^{N\times 1}$ can be chosen to maximize the signal to noise ratio (SNR) of the combined signal $\hat{s}$. One finds for the optimum $\boldsymbol{w} = \boldsymbol{R_n}^{-1}\boldsymbol{h}$, and therefore the signal to noise ratio becomes:

$$\mathrm{SNR} = \frac{\mathrm{E}\left[|\hat{s}|_2^2 \mid \boldsymbol{n} = \boldsymbol{0},\ \boldsymbol{h}\right]}{\mathrm{E}\left[|\hat{s}|_2^2 \mid s = 0\right]} = \sigma_s^2 \boldsymbol{h}^{\mathrm{H}}\boldsymbol{R_n}^{-1}\boldsymbol{h}, \tag{3}$$

where $\sigma_s^2 = \mathrm{E}[|s|^2]$. The diversity measure is then defined as:

$$D = \frac{\left(\mathrm{E}[\mathrm{SNR}]\right)^2}{\mathrm{Var}[\mathrm{SNR}]}. \tag{4}$$

Clearly, the larger the diversity measure D is, the smaller is the relative fluctuation of the SNR around its mean value. Because achieving a small relative fluctuation is the very goal of diversity combining, a larger value of D means a better diversity performance. In this paper, we restrict the discussion to the case of *Rayleigh fading*, such that $\boldsymbol{h}$ contains zero-mean, complex, circularly symmetric Gaussian random entries. With

$$\boldsymbol{\Psi} = \mathrm{const} \cdot \mathrm{E}\left[\boldsymbol{h}\boldsymbol{h}^{\mathrm{H}}\right]\boldsymbol{R_n}^{-1}, \tag{5}$$

where const is an arbitrary non-zero constant, one obtains for the diversity measure the following simple expression:

$$D = D(\boldsymbol{\Psi}) = \frac{(\mathrm{tr}\,\boldsymbol{\Psi})^2}{\mathrm{tr}\,\boldsymbol{\Psi}^2}. \tag{6}$$

It holds true that $1 \le D(\boldsymbol{\Psi}) \le \mathrm{rank}\,\boldsymbol{\Psi}$. Note that $\boldsymbol{\Psi}$ can also be written as:

$$\boldsymbol{\Psi} = \mathrm{const} \cdot \mathrm{E}\left[\boldsymbol{u}\boldsymbol{u}^{\mathrm{H}} \mid \boldsymbol{n} = \boldsymbol{0}\right]\boldsymbol{R_n}^{-1}, \tag{7}$$

i.e., as the (scaled) product of the correlation matrix of a noise-free version of the received signal, and the inverse of the noise correlation matrix.

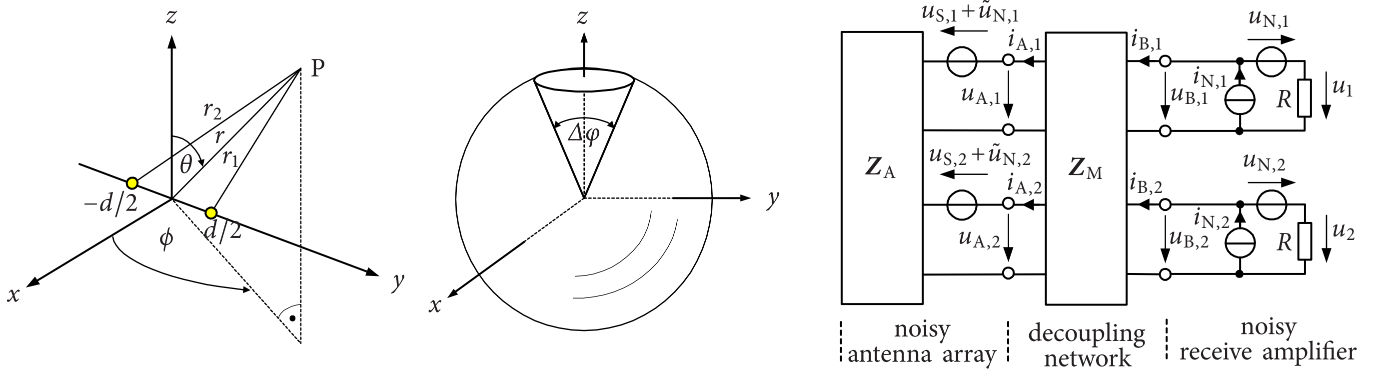

Figure 1: Left: Array composed of two isotropic radiators and a point P in the far-field described by spherical coordinates. Center: Angle-spread of impinging wavefronts. Right: Multiport model of antenna array and decoupling network.

III. System Model

We consider an array of two isotropic radiators placed on the y-axis of a Cartesian coordinate system, centered in its origin and separated by a distance d, as shown on the left hand side of Figure 1. There shall impinge *signal* wavefronts exclusively from within a cone with opening angle $\Delta\varphi$, as shown in the center of Figure 1. The antenna mutual coupling is taken care of by modeling the array as a linear twoport, characterized by its impedance matrix $\boldsymbol{Z}_\mathrm{A} \in \mathbb{C}^{2\times 2}\cdot\Omega$, in series with a voltage source at each port, such that the vector $\boldsymbol{u}_\mathrm{A} = [u_{\mathrm{A},1}\ u_{\mathrm{A},2}]^\mathrm{T}$ of the antenna port voltages equals:

$$\boldsymbol{u}_\mathrm{A} = \boldsymbol{Z}_\mathrm{A}\boldsymbol{i}_\mathrm{A} + \boldsymbol{u}_\mathrm{S} + \tilde{\boldsymbol{u}}_\mathrm{N}, \tag{8}$$

wherein $\boldsymbol{u}_\mathrm{S} = [u_{\mathrm{S},1}\ u_{\mathrm{S},2}]^\mathrm{T}$ is the vector of *open-circuit* voltages induced into the antennas as a result of the impinging *signal* wavefronts, and $\tilde{\boldsymbol{u}}_\mathrm{N} = [\tilde{u}_{\mathrm{N},1}\ \tilde{u}_{\mathrm{N},2}]^\mathrm{T}$ is the vector of *open-circuit noise* voltages received by the antenna, while $\boldsymbol{i}_\mathrm{A} = [i_{\mathrm{A},1}\ i_{\mathrm{A},2}]^\mathrm{T}$ is the vector of antenna port currents. When the background noise impinges on the array *isotropically*, it can be shown that the *open-circuit* antenna noise voltage vector $\tilde{\boldsymbol{u}}_\mathrm{N} \in \mathbb{C}^{2\times 1}\cdot\mathrm{V}$, has the correlation matrix [6]:

$$\mathrm{E}\left[\tilde{\boldsymbol{u}}_\mathrm{N}\tilde{\boldsymbol{u}}_\mathrm{N}^\mathrm{H}\right] = 4\mathrm{k}T_\mathrm{A}\Delta f\,\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}, \tag{9}$$

where k is the Boltzmann constant, T_A is the *noise temperature* of the antenna, and Δf is the bandwidth. The noisy antenna ports are connected to the inputs of a *lossless, reciprocal* matching network, which operation shall be described by:

$$\begin{bmatrix}\boldsymbol{u}_\mathrm{B}\\ \boldsymbol{u}_\mathrm{A}\end{bmatrix} = \boldsymbol{Z}_\mathrm{M}\begin{bmatrix}\boldsymbol{i}_\mathrm{B}\\ -\boldsymbol{i}_\mathrm{A}\end{bmatrix}. \tag{10}$$

Herein, $\boldsymbol{u}_\mathrm{B} = [u_{\mathrm{B},1}\ u_{\mathrm{B},2}]^\mathrm{T}$, and $\boldsymbol{i}_\mathrm{A} = [i_{\mathrm{B},1}\ i_{\mathrm{B},2}]^\mathrm{T}$ are the vectors of the port voltages and port currents at the matching networks' output, as shown on the right hand side of Figure 1. When the impedance matrix $\boldsymbol{Z}_\mathrm{M}$ is chosen as:

$$\boldsymbol{Z}_\mathrm{M} = \begin{bmatrix}\mathrm{j}\,\mathrm{Im}\{Z_\mathrm{out}\}\,\mathbf{I}_2 & \mathrm{j}\sqrt{\mathrm{Re}\{Z_\mathrm{out}\}}\,\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}^{1/2}\\ \mathrm{j}\sqrt{\mathrm{Re}\{Z_\mathrm{out}\}}\,\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}^{1/2} & -\mathrm{j}\,\mathrm{Im}\{\boldsymbol{Z}_\mathrm{A}\}\end{bmatrix}, \tag{11}$$

the output ports of the matching network become electrically *decoupled* and present an output impedance of Z_out. They are connected to two independent noisy receive amplifiers, each of which modeled, in the usual way [7], by its input resistance R, a noise voltage source $u_{\mathrm{N},j}$, and a noise current source $i_{\mathrm{N},j}$, with $j \in \{1,2\}$ and the statistical properties:

$$\left.\begin{aligned}\mathrm{E}\left[\boldsymbol{i}_\mathrm{N}\boldsymbol{i}_\mathrm{N}^\mathrm{H}\right] &= \beta\,\mathbf{I}_2,\\ \mathrm{E}\left[\boldsymbol{u}_\mathrm{N}\boldsymbol{u}_\mathrm{N}^\mathrm{H}\right] &= \beta R_\mathrm{N}^2\,\mathbf{I}_2,\\ \mathrm{E}\left[\boldsymbol{u}_\mathrm{N}\boldsymbol{i}_\mathrm{N}^\mathrm{H}\right] &= \rho\beta R_\mathrm{N}\,\mathbf{I}_2.\end{aligned}\right\} \tag{12}$$

Herein, $\boldsymbol{u}_\mathrm{N} = [u_{\mathrm{N},1}\ u_{\mathrm{N},2}]^\mathrm{T}$, $\boldsymbol{i}_\mathrm{N} = [i_{\mathrm{N},1}\ i_{\mathrm{N},2}]^\mathrm{T}$, while $\beta \in \mathbb{R}_+\cdot\mathrm{A}^2$ is the variance of the receiver noise current within the bandwidth Δf, $R_\mathrm{N} = \sqrt{\mathrm{E}[|u_{\mathrm{N},j}|^2]/\mathrm{E}[|i_{\mathrm{N},j}|^2]}$, is the noise-resistance of the receive amplifiers, while the complex noise correlation equals $\rho = \mathrm{E}[u_{\mathrm{N},j}i_{\mathrm{N},j}^*]/\sqrt{\mathrm{E}[|u_{\mathrm{N},j}|^2]\cdot\mathrm{E}[|i_{\mathrm{N},j}|^2]}$.

The vector $\boldsymbol{u} = [u_1\ u_2]^\mathrm{T}$ contains the two noisy observable output voltages u_1 and u_2, which are shown on the right hand side of Figure 1. From circuit analysis one finds that

$$\boldsymbol{u} = \frac{\mathrm{j}R\sqrt{\mathrm{Re}\{Z_\mathrm{out}\}}}{R+Z_\mathrm{out}}\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}^{-1/2}\boldsymbol{u}_\mathrm{S} + \boldsymbol{n}, \tag{13}$$

where $\boldsymbol{n} \in \mathbb{C}^{2\times 1}\cdot\mathrm{V}$ is the noise voltage with covariance matrix:

$$\boldsymbol{R}_{\boldsymbol{n}} = \mathrm{E}\left[\boldsymbol{n}\boldsymbol{n}^\mathrm{H}\right] = \sigma_n^2\,\mathbf{I}_2. \tag{14}$$

That is, the observed output noise is uncorrelated with identical variance:

$$\sigma_n^2 = \frac{\beta R^2}{|R+Z_\mathrm{out}|^2}\Bigg(|Z_\mathrm{out}|^2 - 2R_\mathrm{N}\,\mathrm{Re}\{\rho^* Z_\mathrm{out}\} + R_\mathrm{N}^2 + \\ + \frac{4\mathrm{k}T_\mathrm{A}\Delta f}{\beta}\,\mathrm{Re}\{Z_\mathrm{out}\}\Bigg), \tag{15}$$

The output impedance Z_out can be chosen such that the signal to noise ratio of u_1 and u_2 is maximized. It turns out that this is achieved when Z_out is set to the value [8]:

$$Z_\mathrm{opt} = R_\mathrm{N}\cdot\left(\sqrt{1-(\mathrm{Im}\{\rho\})^2} + \mathrm{j}\cdot\mathrm{Im}\{\rho\}\right). \tag{16}$$

This makes for

$$\mathrm{E}\left[\boldsymbol{u}\boldsymbol{u}^\mathrm{H}\,|\,\boldsymbol{n}=\mathbf{0}\right]\boldsymbol{R}_{\boldsymbol{n}}^{-1} = \frac{\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}^{-1/2}\,\mathrm{E}\left[\boldsymbol{u}_\mathrm{S}\boldsymbol{u}_\mathrm{S}^\mathrm{H}\right]\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}^{-1/2}}{4\mathrm{k}T_\mathrm{A}\Delta f\,\mathbf{NF}_\mathrm{min}}, \tag{17}$$

where $\mathbf{NF}_\mathrm{min} = 1 + \beta R_\mathrm{N}(\sqrt{1-(\mathrm{Im}\{\rho\})^2} - \mathrm{Re}\{\rho\})/(2\mathrm{k}T_\mathrm{A}\Delta f)$, is the minimum noise figure of the receiver. Note that only the *real-part* of the array impedance matrix $\boldsymbol{Z}_\mathrm{A}$ affects (17). For two *isotropic radiators*, it equals [8], [9]:

$$\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\} = R_\mathrm{r}\boldsymbol{C}, \quad \text{with } \boldsymbol{C} = \begin{bmatrix} 1 & \sin(kd)/(kd) \\ \sin(kd)/(kd) & 1 \end{bmatrix}. \tag{18}$$

Herein R_r is the radiation resistance, and $k = 2\pi/\lambda$, where λ is the wavelength. When we substitute (18) into (17), we can obtain by comparison with (7), the following simple expression for the matrix $\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \boldsymbol{C}^{-1/2}\boldsymbol{\Phi}\,\boldsymbol{C}^{-1/2}, \quad \text{where} \quad \boldsymbol{\Phi} = \mathrm{E}\left[\boldsymbol{u}_\mathrm{S}\boldsymbol{u}_\mathrm{S}^\mathrm{H}\right], \tag{19}$$

provided that we set the non-zero constant in (7) conveniently to the value $4\mathrm{k}T_\mathrm{A}\Delta f R_\mathrm{r}\,\mathrm{NF}_\mathrm{min}$. Notice that $\boldsymbol{\Phi}$ is the correlation matrix of the *open-circuit* array *signal* voltage vector.

IV. Modeling Angle Spread

To proceed further, we now have to compute $\boldsymbol{\Phi}$ as a function of the angle-spread parameter $\Delta\varphi$ (see center of Figure 1), and the antenna separation d. To this end, think of a dipole antenna located in a point P well in the far-field (see left hand side of Figure 1), which excites at the two antennas of the array the electric incident fields:

$$\vec{\boldsymbol{E}}_n^\mathrm{inc} = \vec{\mathbf{e}}^\mathrm{inc}\alpha'\mathrm{e}^{-\mathrm{j}kr_n}/r_n, \tag{20}$$

where $\vec{\mathbf{e}}^\mathrm{inc}$ describes the polarization, α' is proportional to the excitation current of the dipole, and r_n is the distance to the respective antenna, with $n \in \{1,2\}$. In so-called canonical minimum scattering antennas [10], [11], a zero port current leads to zero current density everywhere in the antenna structure, such that no scattered field results. Assuming that the array antennas are of this type, there is only the incident electric field present when the port currents are zero. Therefore, the *open-circuit* voltages can be written as:

$$u_{\mathrm{S},1/2} = \tilde{\alpha}\,\vec{\mathbf{e}}_0\boldsymbol{\cdot}\vec{\boldsymbol{E}}_{1/2}^\mathrm{inc}, \tag{21}$$

where $\tilde{\alpha}$ is an antenna specific constant, and the unit vector $\vec{\mathbf{e}}_0$ describes the polarization of the antennas. With the spherical coordinates θ and ϕ defined on the left hand side of Figure 1, one obtains:

$$r_{1/2} = r \mp \tfrac{1}{2}d\sin(\theta)\sin\phi, \quad \text{for } r \gg d. \tag{22}$$

Thus, by substituting (22) into (20) and the latter into (21):

$$\boldsymbol{u}_\mathrm{S} = \alpha\cdot\boldsymbol{a}(\theta,\phi), \quad \text{where} \quad \boldsymbol{a}(\theta,\phi) = \begin{bmatrix} \mathrm{e}^{\mathrm{j}\frac{1}{2}d\sin(\theta)\sin\phi} \\ \mathrm{e}^{-\mathrm{j}\frac{1}{2}d\sin(\theta)\sin\phi} \end{bmatrix}, \tag{23}$$

and $\alpha = \tilde{\alpha}\alpha'\,\vec{\mathbf{e}}_0\boldsymbol{\cdot}\vec{\mathbf{e}}^\mathrm{inc}\mathrm{e}^{-\mathrm{j}kr}/r$. With multiple transmitting antennas one obtains a linear superposition of the respective individual voltages:

$$\boldsymbol{u}_\mathrm{S} = \sum_n \alpha_n\boldsymbol{a}(\theta_n,\phi_n), \tag{24}$$

where (θ_n,ϕ_n) describes the direction of the n-th remote antenna, and α_n is proportional to its excitation current. Now we model the α_n as mutually uncorrelated random variables, such that $\mathrm{E}[\alpha_n\alpha_m^*] = \gamma_n\delta_{n,m}$, where $\delta_{n,m}$ is unity for $m=n$, and zero else. Hence,

$$\boldsymbol{\Phi} = \mathrm{E}\left[\boldsymbol{u}_\mathrm{S}\boldsymbol{u}_\mathrm{S}^\mathrm{H}\right] = \sum_n \gamma_n\boldsymbol{a}(\theta_n,\phi_n)\boldsymbol{a}^\mathrm{H}(\theta_n,\phi_n). \tag{25}$$

In the limit of infinitely many sources, one obtains the correlation matrix of a continuous distribution of impinging waves:

$$\boldsymbol{\Phi} = \mathrm{E}\left[\boldsymbol{u}_\mathrm{S}\boldsymbol{u}_\mathrm{S}^\mathrm{H}\right] = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\gamma(\theta,\phi)\boldsymbol{a}(\theta,\phi)\boldsymbol{a}^\mathrm{H}(\theta,\phi)\,\mathrm{d}\theta\mathrm{d}\phi. \tag{26}$$

Because

$$\boldsymbol{\Phi}_{n,n} = \mathrm{E}\left[|u_{\mathrm{S},n}|^2\right] = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\gamma(\theta,\phi)\,\mathrm{d}\theta\mathrm{d}\phi, \tag{27}$$

we can appreciate that the term $\gamma(\theta,\phi)\mathrm{d}\theta\mathrm{d}\phi$ is proportional to the power that can be received from the directional window between θ and $\theta+\mathrm{d}\theta$, and ϕ and $\phi+\mathrm{d}\phi$. Writing this same power as the product of the *power density* (power per unit area) $P'(\theta,\phi)$, and the infinitesimal area $\mathrm{d}A = r_0^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi$, it follows that

$$\gamma(\theta,\phi) \sim P'(\theta,\phi)r_0^2\sin(\theta). \tag{28}$$

Herein, r_0 is a *fixed* radius of a sphere around the array, at which surface the power density $P'(\theta,\phi)$ is to be evaluated. Hence,

$$\boldsymbol{\Phi} = \mathrm{const}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}P'(\theta,\phi)\boldsymbol{a}(\theta,\phi)\boldsymbol{a}^\mathrm{H}(\theta,\phi)\sin(\theta)\,\mathrm{d}\theta\mathrm{d}\phi. \tag{29}$$

Referring to the center of Figure 1, the power density $P'(\theta,\phi)$ shall be *constant* for all $0 \le \theta \le \Delta\varphi/2$ and all $0 \le \phi \le 2\pi$, and *zero* elsewhere. In this way, one obtains from (29):

$$\boldsymbol{\Phi} = \sigma_\mathrm{S}^2\begin{bmatrix} 1 & \rho_\mathrm{S} \\ \rho_\mathrm{S} & 1 \end{bmatrix}, \quad \text{where } \rho_\mathrm{S} = \frac{\int_0^{\Delta\varphi/2}\mathrm{J}_0(kd\sin\theta)\sin(\theta)\,\mathrm{d}\theta}{2\sin^2(\Delta\varphi/4)}. \tag{30}$$

The term σ_S^2 denotes the variance of the array's *open-circuit* signal voltages, and ρ_S is their correlation coefficient, while $\mathrm{J}_0(\cdot)$ is the Bessel function of the first kind and zero-th order. Notice that $\lim_{kd\to 0}(\rho_\mathrm{S}) = 1$, that is, the open-circuit array voltages become *coherent* as the electric distance $kd = 2\pi d/\lambda$ is reduced towards zero. This is the reason behind the widespread belief that compact antenna arrays have poor diversity performance. This conjecture is, however, not justified because it is not the $\boldsymbol{\Phi}$ matrix which tells about the diversity performance, but rather the matrix $\boldsymbol{\Psi}$ given in (19).

V. Diversity of Compact Arrays

To judge the diversity performance of the array we have to look at the correlation matrix $\boldsymbol{\Psi}$ from (19). Substituting (30) into (19), one obtains with the help of (18) and

$$\begin{bmatrix} 1 & \Theta \\ \Theta & 1 \end{bmatrix}^{-1/2}\begin{bmatrix} 1 & \rho_\mathrm{S} \\ \rho_\mathrm{S} & 1 \end{bmatrix}\begin{bmatrix} 1 & \Theta \\ \Theta & 1 \end{bmatrix}^{-1/2} = \frac{1}{1-\Theta^2}\begin{bmatrix} 1-\rho_\mathrm{S}\Theta & \rho_\mathrm{S}-\Theta \\ \rho_\mathrm{S}-\Theta & 1-\rho_\mathrm{S}\Theta \end{bmatrix}, \tag{31}$$

the following simple expression for $\boldsymbol{\Psi}$:

$$\boldsymbol{\Psi} = \sigma^2\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \quad \text{where } \rho = \frac{\rho_\mathrm{S} - j_0(kd)}{1-\rho_\mathrm{S}j_0(kd)} \tag{32}$$

is the relevant correlation coefficient. Herein, we have introduced $j_0(x) = \sin(x)/x$ for notational convenience. The variance $\sigma^2 = \sigma_\mathrm{S}^2\cdot(1-\rho_\mathrm{S}j_0(kd))\,/\,\left(1-j_0^2(kd)\right)$.

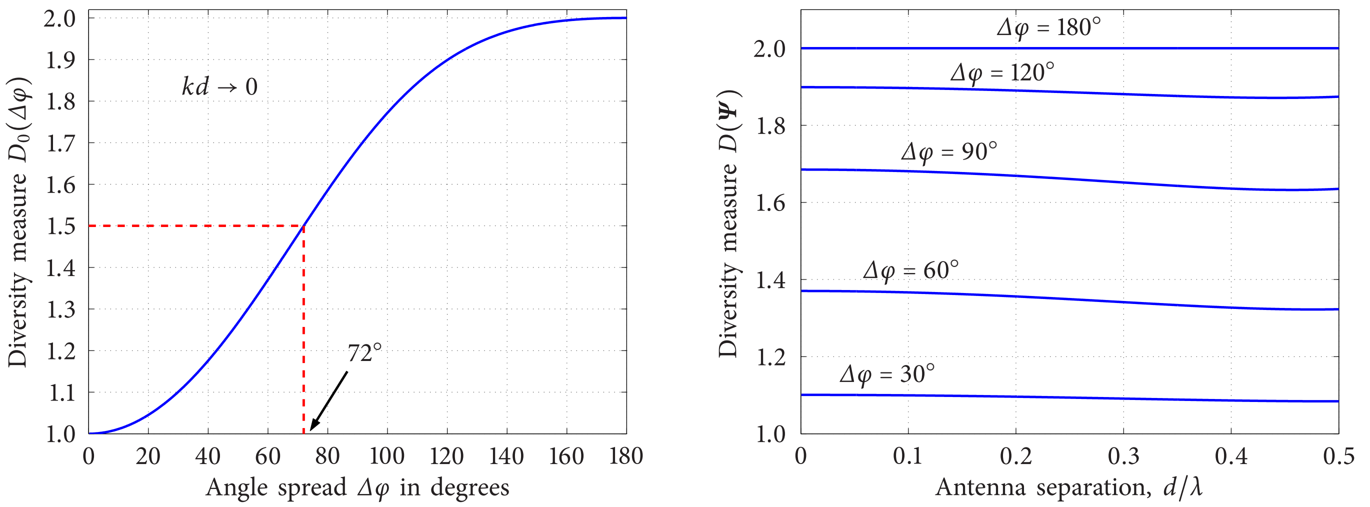

Figure 2: LEFT: Diversity measure $D_0(\Delta\varphi)$ of the infinitesimal array ($kd \to 0$) as function of angle spread. RIGHT: Diversity measure $D(\boldsymbol{\Psi})$ as function of antenna separation for different angle spreads.

A. Supercompact Arrays

Let us look into the extreme case of the *infinitesimal array*, where $kd \to 0$. Substituting the expression for ρ_S from (30) into (32) one obtains:

$$\boldsymbol{\Psi}_0 = \lim_{kd \to 0} \boldsymbol{\Psi} = \sigma_0^2 \begin{bmatrix} 1 & \rho_0 \\ \rho_0 & 1 \end{bmatrix}, \tag{33}$$

where

$$\rho_0 = \frac{4\cos^2(\Delta\varphi/4)\cos(\Delta\varphi/2)}{7 - 2\cos(\Delta\varphi/2) - \cos\Delta\varphi} \tag{34}$$

is the correlation coefficient for the infinitesimal array, and the variance $\sigma_0^2 = \sigma_S^2 \cdot (7 - 2\cos(\Delta\varphi/2) - \cos\Delta\varphi)/8 \geq \sigma_S^2/2$. When we put (34) into (33) and the latter into (6), the diversity measure for the *infinitesimal array* becomes:

$$D_0(\Delta\varphi) = \lim_{kd \to 0} D(\boldsymbol{\Psi}) = D(\boldsymbol{\Psi}_0) =$$

$$= \frac{2\,(7 - 2\cos(\Delta\varphi/2) - \cos\Delta\varphi)^2}{55 - 20\cos(\Delta\varphi/2) - 8\cos(\Delta\varphi) + 4\cos(3\Delta\varphi/2) + \cos 2\Delta\varphi}, \tag{35}$$

which is shown in graphical from on the left hand side of Figure 2. Note that $D_0(\Delta\varphi)$ is strictly increasing with increasing angle spread. For a moderately large angle spread of $\Delta\varphi \approx 72°$ the diversity measure has climbed to a fairly large value of 1.5. As the angle spread is increased further, the diversity measure raises towards its *maximum value* of 2, which it *achieves* for $\Delta\varphi = 180°$.

B. Compact Arrays

While the limit $kd \to 0$ is interesting from a theory point of view, the cases of small but finite kd are more interesting in practice. The right hand side of Figure 2 shows the diversity measure $D(\boldsymbol{\Psi})$ as a function of antenna separation d/λ for a number of different angle spreads. Clearly, the diversity performance is *essentially unchanged* as the antenna separation is reduced from half a wavelength towards zero. In fact, the diversity measure even slightly increases. Note that for an angle-spread of $\Delta\varphi = 180°$, the diversity measure exactly equals 2 (its maximum value) independent of the separation d/λ.

VI. Conclusion

We have found that compact antenna arrays with less than half a wavelength separation between neighboring antennas are perfectly able to deliver excellent diversity performance. This comes about because of the joint effect of mutual antenna coupling in conjunction with a properly designed decoupling multiport which is connected between the antenna ports and the receivers. With isotropic background antenna noise, and in the absence of heat loss, the diversity performance is limited only by the angle spread of the impinging wavefronts but *not* by the antenna separation inside the array.

References

[1] J. C. Liberti and T. S. Rappaport, *Smart Antennas for Wireless Communications*. New Jersey: Prentice Hall, 1999.

[2] D. S. Shiu, G. J. Foschini, M. J. Gans, and J. M. Kahn, “Fading correlation, and its effect on the capacity of multi-element antenna systems,” *IEEE Transactions on Communications*, vol. 48(3), pp. 502–512, 2000.

[3] J. S. Hammerschmidt and C. Brunner, “The implications of Array and Multipath Geometries in Spatial Processing,” in *Proc. IEEE/IEE Int. Conf. on Telecommunications*, Acapuloco, Mexico, May 2000.

[4] D. Eßer and H. J. Chaloupka, “Design Approach for a Class of Compact Multiport Antennas,” in *Proc. IEEE 2005 Int. Symp. on Microwave, Antenna, Propagat. and EMC Technol. for Wireless Commun. (MAPE 2005)*, 2005, pp. 144–147.

[5] M. T. Ivrlač and J. A. Nossek, “Quantifying Diversity and Correlation in Rayleigh Fading MIMO Communication Systems,” in *Proc. 3rd. International Symposium on Signal Processing and Information Theory (ISSPIT)*, Darmstadt, Germany, 2003.

[6] R. Q. Twiss, “Nyquist’s and Thevenin’s Theorems Generalized for Nonreciprocal Linear Networks,” *Journal of Applied Physics*, vol. 26, pp. 599–602, May 1955.

[7] H. Hillbrand and P. Russer, “An Efficient Method for Computer Aided Noise Analysis of Linear Amplifier Networks,” *IEEE Transactions on Circuits and Systems*, vol. 23(4), pp. 235–238, jun 1976.

[8] M. T. Ivrlač and J. A. Nossek, “Toward a Circuit Theory of Communication,” *IEEE Transactions on Circuits and Systems I: Regular Papers*, vol. 57(7), pp. 1663–1683, 2010.

[9] A. I. Uzkov, “An Approach to the Problem of Optimum Directive Antennae Design,” *Comptes Rendus (Doklady) de l’Académie des Sciences de l’URSS*, vol. 53(1), pp. 35–38, 1946.

[10] K. Kahn and H. Kurss, “Minimum-Scattering Antennas,” *IEEE Transactions on Antennas and Propagation*, vol. AP-13, pp. 671–675, sep 1965.

[11] W. Wasylkiwskyj and W. K. Kahn, “Theory of Mutual Coupling Among Minimum-Scattering Antennas,” *IEEE Transactions on Antennas and Propagation*, vol. 18(2), pp. 204–216, mar 1970.