Scalar diffraction problem for N arbitrarily shaped surfaces of revolution: rigorous approach

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Abstract

This paper describes the application of the analytical regularization method to the solution of scalar diffraction problems. Reflectors composed from several coaxial surfaces of revolution are considered. The surfaces are arbitrarily shaped and may be closed or open. The initial problem is transformed to an infinite linear algebraic system of second kind in l_2 space. Numerical solutions to such systems with a guaranteed pre-specified accuracy are readily obtained.

1. Introduction

We derive a mathematically rigorous and numerically stable solution to the above problem by the Method of Analytical Regularization (MoR), a semi-analytical semi-numerical technique. Initially this approach was developed for rigorous analysis of wave scattering by the screens of canonical form, either open or closed: see, for example, [1, 2]. Later, accurate solutions to scattering by infinitely long cylinders of arbitrary cross-section [3] were obtained by MoR. The extension of MoR to wave diffraction by a screen in the form of surface of revolution with arbitrary profile was suggested in [4] and later developed in [5] to solve the potential problem for a single charged surface of revolution with arbitrary profile. Then the technique was generalized to consider several charged coaxial surfaces of revolution [6]. The MoR was also successfully extended to the solution of a scalar problem for a single surface of revolution of arbitrary profile illuminated by a time-harmonic wave [7]. This paper presents the extension of the approach in [6] to scalar diffraction by N screens of arbitrary profile. Many practical and important scatterers can be modeled as a number of symmetric screens of revolution with the desired smooth profile. This paper develops the basis for the algorithm representing numerical analysis of scattering by such assemblies of screens.

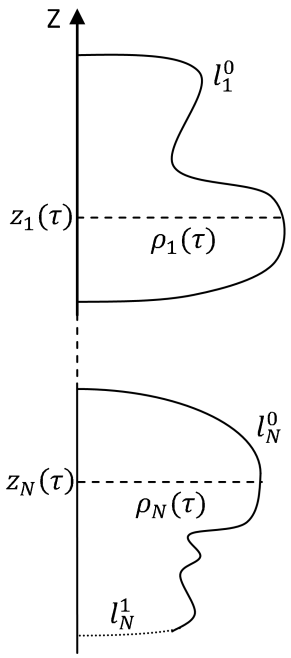

Fig. 1

2. Problem Formulation

The geometry of the problem is shown in Fig. 1. We consider $N \geq 1$ arbitrarily shaped (either closed or open) coaxial surfaces of revolution $S_1^0,...,S_N^0$. We will use a cylindrical coordinate system $\{\rho,\varphi,z\}$ with z-axis coincident with the axis of symmetry of the surfaces. For each $n=1...N$ the surface S_n^0 is formed by the rotation of a generating curve l_n^0 about the z-axis. Each curve l_n^0 is assumed to be a connected part of a smooth and non-self-crossing curve l_n, the ends of which are placed on the z-axis, and which lies in the plane $\varphi=0$. The surface S_n, obtained by the rotation of the curve l_n, is assumed to be smoothly isomorphic (diffeomorphic) to a spherical surface.

Let $\{\rho_n(\tau),0,z_n(\tau)\}$ be a parameterization of the curve l_n, with a parameter $\tau\in[0,\pi]$. The parameterizations are assumed to satisfy the following conditions:

1) $z_n,\rho_n\in C^2(0,\pi)$

2) $\rho_n(0)=\rho_n(\pi)=0,\quad z_n'(0)=z_n'(\pi)=0$

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3) $0 < \ell_n(\tau) = \sqrt{\{\rho_n'(\tau)\}^2 + \{z_n'(\tau)\}^2} < \infty, \quad \tau \in [0,\pi]$

We can choose τ_n^0 so that the contour l_n^0 corresponds to the interval $\tau \in \left[0, \tau_n^0\right]$ in the parameterization of l_n introduced above. Thus each surface S_n^0 is parameterized in cylindrical coordinates by $\{\rho_n(\tau), \varphi, z_n(\tau)\}$, where $\tau \in \left[0, \tau_n^0\right]$ and $\varphi \in [-\pi, \pi)$.

Let the screens be illuminated by a time-harmonic wave $U_0(q,t) = U(q)e^{-i\omega t}$, where ω is the angular frequency of the incident field. We consider the Dirichlet diffraction problem: we seek the scattered field $U^{sc}(q)$ which satisfies the homogeneous Helmholtz equation

$$\left(\Delta + k^2\right) U^{sc}(q) = 0, \quad q \in \mathbb{R}^3 \setminus \bigcup_{n=1}^{N} S_n^0, \tag{1}$$

subject to the Dirichlet boundary condition

$$U^{sc}(q) + U(q) = 0, \quad q \in \bigcup_{n=1}^{N} S_n^0, \tag{2}$$

where *k* is the (free-space) wave number. Moreover, the diffracted field must satisfy the Sommerfeld radiation condition

$$\lim_{|q|\to\infty} |q| \left(\frac{\partial}{\partial |q|} - ik \right) U^{sc}(q) = 0 \tag{3}$$

and the condition on the finiteness of energy within any bounded volume V of space (including the vicinity of edges):

$$\iiint_V dV \left\{ \left|\nabla U^{sc}\right|^2 + k^2 \left|U^{sc}\right|^2 \right\} < \infty. \tag{4}$$

3. Solution

The field scattered by each surface S_n^0 can be represented in the following integral form using the unknown scalar current density $\Sigma_n(p)$:

$$U_n^{sc}(q) = \iint_{S_n^0} \Sigma_n(p) G(q,p)\, ds_p, \quad q \in \mathbb{R}^3, p \in S_n^0, \tag{5}$$

where $G(q,p) = e^{ik|p-q|} / \left(4\pi |p-q|\right)$ is the Green's function for Helmholtz equation (1) in three-dimensional free space. Using the superposition principle we express the total field $U^{sc}(q)$ as the sum of N fields $U_n^{sc}(q)$:

$$U^{sc}(q) = \sum_{n=1}^{N} \iint_{S_n^0} \Sigma_n(p) G(q,p)\, ds_p. \tag{6}$$

Let $p \in S_n$, $q \in S_m (1 \le n, m \le N)$, then $p = \{\rho_n(\tau_p), \varphi_p, z_n(\tau_p)\}$, $q = \{\rho_m(\tau_q), \varphi_q, z_m(\tau_q)\}$ for some $\tau_p, \tau_q \in [0,\pi]$ and $\varphi_p, \varphi_q \in [0, 2\pi)$. It is easy to see that the Green's function is a function of difference of the azimuthal angles: $G(q,p) = G_{mn}(\tau_q, \tau_p, \varphi_q - \varphi_p)$. Applying boundary conditions (2) to the total potential (6) and transforming the surface integrals we obtain the following system of functional equations:

$$\sum_{n=1}^{N}\int_{0}^{\tau_n^0}\int_{-\pi}^{\pi}\sigma_n(\tau_p,\varphi_p)G_{mn}(\tau_q,\tau_p,\varphi_q-\varphi_p)\rho_n(\tau_p)\ell_n(\tau_p)d\varphi_p d\tau_p=-U_m(\tau_q,\varphi_q),\quad m=1..N \tag{7}$$

where $\sigma_n(\tau,\varphi)=\Sigma_n\left(\{\rho_n(\tau),\varphi,z_n(\tau)\}\right)$ and $U_m(\tau,\varphi)=U\left(\{\rho_m(\tau),\varphi,z_m(\tau)\}\right)$.

For each $n=1...N$ we introduce a new unknown function $X_n(\tau,\varphi)=-\xi_n(\tau)\ell_n(\tau)\sigma_n(\tau,\varphi)$, describing the normalized scalar current density, where $\xi_n(\tau)=\sqrt{\rho_n(\tau)/\sin\tau}$, and extend it to the whole surface S_n, by defining it to be zero on the aperture section of this surface. Using azimuthal Fourier series expansions of the functions X_n, U_m and G_{mn}, defined for each function f by

$$f(\tau,\varphi)=\sum_{j=-\infty}^{\infty}f^j(\tau)e^{ij\varphi},\quad f^j(\tau)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\tau,\varphi)e^{-ij\varphi}d\varphi\quad , \tag{8}$$

we rewrite the system (7) in the following form

$$\sum_{n=1}^{N}\int_{0}^{\pi}F_{mn}^j(\tau_q,\tau_p)X_n^j(\tau_p)\sin\tau_p d\tau_p=\frac{1}{2\pi}\xi_m(\tau_q)U_m^j(\tau_q),\quad m=1...N,\ j\in\mathbb{Z}, \tag{9}$$

where $F_{mn}^j(\tau_q,\tau_p)=\xi_m(\tau_q)\xi_n(\tau_p)G_{mn}^j(\tau_q,\tau_p)$. In what follows, let us fix the index j and consider equations corresponding to its value separately.

Obviously, for $m\neq n$, the functions F_{mn}^j are smooth, but the functions F_{nn}^j have singularities when $\tau_q-\tau_p\to 0$ and $\tau_q,\tau_p\to 0,\pi$ for all $n=1...N$. On the basis of detailed investigation of their singular behavior we introduce the splitting $F_{nn}^j=F^j+D_n^j$, where F^j is a special case of functions F_{nn}^j for Laplace equation (i.e. with $k=0$ in (6)) for the surface of the unit sphere, which is parameterized in standard way $\{\sin\tau,\varphi,\cos\tau\}$, and D_n^j is the remainder function. Below we will use the well-known diagonal presentation (see [1]) for F^j

$$F^j(\tau_q,\tau_p)=\frac{1}{4\pi}\sum_{l=|j|}^{\infty}\frac{1}{\varsigma_l}\hat{P}_l^{|j|}(\cos\tau_q)\hat{P}_l^{|j|}(\cos\tau_p), \tag{10}$$

where $\varsigma_l=l+1/2$ and $\hat{P}_l^{|j|}(x)$ is the normalized associated Legendre polynomial of degree $|j|$ and order l. We will also use the most general form of Fourier-Legendre expansion for the functions F_{mn}^j, D_n^j, X_n^j, $\xi_m U_m^j$:

$$\begin{cases}
F_{mn}^j(\tau_q,\tau_p)=\sum_{l=|j|}^{\infty}\sum_{s=|j|}^{\infty}F_{mn}^{jls}\hat{P}_l^{|j|}(\cos\tau_q)\hat{P}_s^{|j|}(\cos\tau_p),\quad m\neq n\\
D_n^j(\tau_q,\tau_p)=\sum_{l=|j|}^{\infty}\sum_{s=|j|}^{\infty}D_n^{jls}\hat{P}_l^{|j|}(\cos\tau_q)\hat{P}_s^{|j|}(\cos\tau_p),\\
X_n^j(\tau_p)=\sum_{l=|j|}^{\infty}x_n^{jl}\hat{P}_l^{|j|}(\cos\tau_p),\\
\xi_m(\tau_q)U_m^j(\tau_q)=\sum_{l=|j|}^{\infty}u_m^{jl}\hat{P}_l^{|j|}(\cos\tau_q)
\end{cases} \tag{11}$$

Due to the smoothness of the functions F_{mn}^j, D_n^j and $\xi_m U_m^j$ a reliable and efficient calculation procedure for their Fourier-Legendre coefficients can be constructed. It can be also proved that

$$\sum_{l=|j|}^{\infty}\sum_{l=|j|}^{\infty}\left(\varsigma_l\varsigma_s\right)^2\left|F_{mn}^{jls}\right|^2<\infty \text{ and } \sum_{l=|j|}^{\infty}\sum_{l=|j|}^{\infty}\left(\varsigma_l\varsigma_s\right)^2\left|D_{n}^{jls}\right|^2<\infty. \tag{12}$$

After substituting expansions (10) and (11) into the system (9) and subsequent term-by-term integration with respect to the parameter τ_p on the whole interval $[0,\pi]$, the following system of functional equations can be obtained:

$$\begin{cases}\sum\limits_{\substack{n=1\\ n\neq m}}^{N}\sum\limits_{l=|j|}^{\infty}\sum\limits_{s=|j|}^{\infty}F_{mn}^{jls}x_n^{js}\hat{P}_l^{|j|}(\cos\tau)+\dfrac{1}{4\pi}\sum\limits_{l=|j|}^{\infty}\varsigma_l^{-1}x_m^{jl}\hat{P}_l^{|j|}(\cos\tau)+\sum\limits_{l=|j|}^{\infty}\sum\limits_{s=|j|}^{\infty}D_m^{jls}x_m^{js}\hat{P}_l^{|j|}(\cos\tau)=\dfrac{1}{2\pi}\sum\limits_{l=|j|}^{\infty}u_m^{jl}\hat{P}_l^{|j|}(\cos\tau), & \tau\in\left[0,\tau_m^0\right]\\ \sum\limits_{l=|j|}^{\infty}x_m^{jl}\hat{P}_l^{|j|}(\cos\tau)=0, & \tau\in\left(\tau_m^0,\pi\right]\end{cases} \tag{13}$$

Using the well-known relation between the normalized Legendre functions and Jacobi polynomials $\hat{P}_l^{|j|}(\cos\tau)=(-1)^{|j|}\sin^{|j|}\tau\hat{P}_{l-|j|}^{(|j|,|j|)}(\cos\tau)$, we can rearrange this system in the form

$$\begin{cases}\sum\limits_{l=0}^{\infty}\varsigma_{l+|j|}^{1/2}y_{lN+m-1}^{j}\hat{P}_l^{(|j|,|j|)}(z)=0, & z\in\left[-1,z_m^0\right)\\ \sum\limits_{l=0}^{\infty}\varsigma_{l+|j|}^{-1/2}\left\{y_{lN+m-1}^{j}+\sum\limits_{s=0}^{\infty}H_{lN+m-1,s}^{j}y_s^j-U_{lN+m-1}^{j}\right\}\hat{P}_l^{(|j|,|j|)}(z)=0, & z\in\left[z_m^0,1\right]\end{cases} \tag{14}$$

where $y_{(s-|j|)N+n-1}^{j}=\varsigma_s^{-1/2}x_n^{js}$, $H_{(l-|j|)N+m-1,(s-|j|)N+n-1}^{j}=4\pi\varsigma_l^{1/2}\varsigma_s^{1/2}F_{mn}^{jls}$, $H_{(l-|j|)N+m-1,(s-|j|)N+m-1}^{j}=4\pi\varsigma_l^{1/2}\varsigma_s^{1/2}D_m^{jls}$ and $U_{(l-|j|)N+m-1}^{j}=2\varsigma_l^{1/2}u_m^{jl}$. The last system can be transformed to a well-conditioned infinite system of algebraic equations of the second kind by the method of Abel's integral transform (see [2] for details). The details of the transformation are lengthy but are similar to that of [7], and are omitted. Numerical solutions to such systems are readily obtained by truncation of the infinite system to a finite one in a way that ensures a guaranteed pre-specified accuracy. Numerical results illustrating the accuracy and efficiency of our algorithm for a variety of configurations will be presented.

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