Temporal Evolution of the Irrotational and Solenoidal Cavity Modes

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Abstract

An outline of the evolutionary approach to time-domain electromagnetics is presented in a compact form available for practical using. A cavity is loaded with a given source of transient signal. The cavity field is presented via expansions in terms of the solenoidal and irrotational modes having time-dependent modal amplitudes. The differential equations with time derivative are derived from Maxwell's equations for the amplitudes jointly with appropriate initial conditions. The frequency-domain theory usually interprets the irrotational modes as some static fields. Graphical results illustrating the time dependence of the irrotational and solenoidal modes will be exhibited in the presentation.

1 Introduction

Modern powerful computers and effective computational methods operating via finite-difference time-domain procedures are capable of providing scientists with huge volume of numerical data and/or computer graphics for electromagnetic fields in the time domain. Theoretical methods offer us the means for interpreting the data, to search for cause and effect relationships in phenomena, and to attain physical understanding and insight, ultimately.

In '80s, a new Evolutionary Approach to time-domain Electromagnetics (EAE) was proposed as an alternative to the classical time-harmonic field theory aimed on the frequency domain [1]. Several examples of implementation of the EAE in studies of temporal cavity oscillations are listed, for clarity's sake, in [2-4]. In this presentation, a compact scheme of the approach is presented along with practical recommendations for applications.

2 Outline of the Evolutionary Approach to Electromagnetics

2.1 Extraction of a Self-Adjoint Operator from Maxwell's Equations

Introduce a *real-valued* six-component "electromagnetic" time-domain field vector $\mathcal{X}(\mathbf{r},t)$ composed as

$$\mathcal{X}(\mathbf{r},t) = \mathrm{col}\left(\mathcal{E}(\mathbf{r},t), \mathcal{H}(\mathbf{r},t)\right) \tag{1}$$

where $\mathbf{r}$ is a position vector of a point of observation, t is an observation time, $\mathcal{E}$ and $\mathcal{H}$ are the field vectors, col means *column.* Then, the Maxwell's vector equations can be written as a six-component equation

$$\mathfrak{R}\mathcal{X}(\mathbf{r},t) = \begin{pmatrix} \partial_t \mathcal{E}(\mathbf{r},t) + (\sigma/\epsilon_0)\,\mathcal{E}(\mathbf{r},t) + \epsilon_0^{-1}\mathcal{J}_e(\mathbf{r},t) \\ -\partial_t \mathcal{H}(\mathbf{r},t) - \mu_0^{-1}\mathcal{J}_h(\mathbf{r},t) \end{pmatrix} \tag{2}$$

where σ is a conductivity intended for modelling possible losses in the cavity volume V, $\mathcal{J}_e$ and $\mathcal{J}_h$ are *given* functions of external impressed sources of electric and magnetic kind, respectively, $\mathfrak{R}$ is a *self-adjoint* operator which is composed of aggregation of a 6×6 matrix *differential procedure,* $\mathfrak{R}'$, specified as

$$\mathfrak{R}'\mathcal{X}(\mathbf{r},t) = \begin{pmatrix} \mathcal{O} & \epsilon_0^{-1}\nabla\times \\ \mu_0^{-1}\nabla\times & \mathcal{O} \end{pmatrix}\begin{pmatrix} \mathcal{E}(\mathbf{r},t) \\ \mathcal{H}(\mathbf{r},t) \end{pmatrix}, \quad \text{where} \quad \mathcal{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3}$$

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and the algebraic *boundary conditions,* $[\mathbf{n}\times\mathcal{E}]\,|_S=0$ and $(\mathbf{n}\cdot\mathcal{H})\,|_S=0$, where $\mathbf{n}$ is a unit vector outward normal to a perfectly conducting closed singly connected cavity surface, S. Finally, the operator $\mathfrak{R}$ is

$$\mathfrak{R}\mathcal{X}(\mathbf{r},t)=\begin{cases}\mathfrak{R}'\mathcal{X}(\mathbf{r},t)\,, & \mathbf{r}\in V,\ \mathbf{r}\notin S\,,\\ \mathbf{n}\times\mathcal{E}(\mathbf{r},t)=0,\ \mathbf{n}\cdot\mathcal{H}(\mathbf{r},t)=0, & \mathbf{r}\notin V,\ \mathbf{r}\in S\,.\end{cases}\tag{4}$$

The observation of (3) and (4) shows that the operator $\mathfrak{R}$ acts on the space variables, $\mathbf{r}$, solely, meanwhile the time variable, t, plays role of a parameter. This fact suggests to introduce a *functional space* of solutions with its elements, $\mathfrak{X}(\mathbf{r})$, composed as the six-component vectors

$$\mathfrak{X}(\mathbf{r})=\mathrm{col}\,(\mathbf{E}(\mathbf{r})\,,\mathbf{H}(\mathbf{r}))\,;\quad [\mathbf{n}\times\mathbf{E}]_S=\mathbf{0},\quad (\mathbf{n}\cdot\mathbf{H})\,|_S=0\tag{5}$$

where the vector functions $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ are *real-valued,* twice differentiable within the domain V and subjected to the same boundary conditions over the boundary S as in (4). Specify the *space of solutions* by introducing an inner product for any pair of the vectors $\mathfrak{X}_1=\mathrm{col}\,(\mathbf{E}_1,\mathbf{H}_1)$ and $\mathfrak{X}_2=\mathrm{col}\,(\mathbf{E}_2,\mathbf{H}_2)$ as

$$\langle\mathfrak{X}_1(\mathbf{r})\,,\mathfrak{X}_2(\mathbf{r})\rangle=\frac{1}{V}\int_V(\epsilon_0\mathbf{E}_1(\mathbf{r})\cdot\mathbf{E}_2(\mathbf{r})+\mu_0\mathbf{H}_1(\mathbf{r})\cdot\mathbf{H}_2(\mathbf{r}))\;dv.\tag{6}$$

Evidently, we deal with Hilbert space, $L_2(V)$, of the vector functions varying within the closed domain V.

The *difference* of any pair of the following *bilinear forms* (inner products) holds as

$$\langle\mathfrak{R}\,\mathfrak{X}_1(\mathbf{r})\,,\mathfrak{X}_2(\mathbf{r})\rangle-\langle\mathfrak{X}_1(\mathbf{r})\,,\mathfrak{R}\,\mathfrak{X}_2(\mathbf{r})\rangle=0.\tag{7}$$

This fact signifies that the operator $\mathfrak{R}$ is *self-adjoint* and the operator eigenvalue equation holds as

$$\mathfrak{R}\,\mathfrak{X}_n(\mathbf{r})=\omega_n\,\mathfrak{X}_n(\mathbf{r}):\qquad \mathbf{r}\in V,\ \mathbf{r}\in S\tag{8}$$

where ω_n's are the real-valued eigenvalues, subscript $n=0,\pm1,\pm2,\ldots$ specifies the distribution of eigenvalues on a real axis in increasing order of their numerical values, and $\mathfrak{X}_n=\mathrm{col}\,(\mathbf{E}_n,\mathbf{H}_n)$ are the eigenvectors corresponding to these eigenvalues. The spectrum $\{\omega_n\}$ is discrete because the domain V is finite.

2.2 Complete Set of the Solenoidal and Irrotational Cavity Modes

Substitution of the operator $\mathfrak{R}$ in equation (8) results in an equivalent boundary eigenvalue problem as

$$\begin{cases}\nabla\times\mathbf{H}_n(\mathbf{r})=\omega_n\,\epsilon_0\mathbf{E}_n\;(\mathbf{r})\,, & (\mathbf{n}\cdot\mathbf{H}_n(\mathbf{r}))\,|_S=0\\ \nabla\times\mathbf{E}_n(\mathbf{r})=\omega_n\,\mu_0\mathbf{H}_n\;(\mathbf{r})\,, & [\mathbf{n}\times\mathbf{E}_n(\mathbf{r})]\,|_S=0\end{cases}\tag{9}$$

written in terms of the constituents of the eigenvectors $\mathfrak{X}_n=\mathrm{col}\,(\mathbf{E}_n,\mathbf{H}_n)$. Analysis of problem (9) ascertains that it yields *six* varieties of the solutions originating six *subspaces* in the space of solutions. The solutions corresponding to the eigenvalues $\omega_n\neq0$ yield *four* subspaces of the *solenoidal* vectors. If the cavity volume is a short-circuited part of a cylinder, the solenoidal vectors can be associated with the $TE-$ and $TM-$ cavity modes, physically. They are the solutions to the following boundary eigenvalue problems for Laplacian, ∇^2 :

$$\boxed{\boxed{\mathfrak{E}'\{\mathbf{E}_n'\}:}\quad\left\{\ \nabla^2\,\mathbf{E}_n'(\mathbf{r})+\left(\omega_n'^2c^{-2}\right)\mathbf{E}_n'(\mathbf{r})=\mathbf{0},\quad \nabla\cdot\mathbf{E}_n'(\mathbf{r})=0,\quad [\mathbf{n}\times\mathbf{E}_n'(\mathbf{r})]_S=\mathbf{0}\ \right\}_{n=1}^{\infty}}$$

$$\boxed{\boxed{\mathfrak{H}'\{\mathbf{H}_n'\}:}\quad\{\mathbf{H}_n'(\mathbf{r})=\nabla\times\mathbf{E}_n'(\mathbf{r})\,/\,(\omega_n'\,\mu_0)\}_{n=1}^{\infty}}\ \ \nabla\cdot\mathbf{H}_n'(\mathbf{r})=0,\quad \left(\mathbf{n}\cdot\mathbf{H}_n'(\mathbf{r})\right)|_S=0\tag{10}$$

where $\omega_n'>0$ are the real-valued eigenvalues of the operator $\mathfrak{R}$, $n=1,2,\ldots,\,c^{-2}=\epsilon_0\mu_0$; and

$$\boxed{\boxed{\mathfrak{H}''\{\mathbf{H}_n''\}:}\quad\left\{\ \nabla^2\,\mathbf{H}_n''(\mathbf{r})+\left(\omega_n''^2c^{-2}\right)\mathbf{H}_n''(\mathbf{r})=\mathbf{0},\quad \nabla\cdot\mathbf{H}_n''(\mathbf{r})=0,\quad \left(\mathbf{n}\cdot\mathbf{H}_n''(\mathbf{r})\right)|_S=0\ \right\}_{n=1}^{\infty}}$$

$$\boxed{\boxed{\mathfrak{E}''\{\mathbf{E}_n''\}:}\quad\{\mathbf{E}_n''(\mathbf{r})=\nabla\times\mathbf{H}_n''(\mathbf{r})\,/\,(\omega_n''\,\epsilon_0)\}_{n=1}^{\infty}}\ \ \nabla\cdot\mathbf{E}_n''(\mathbf{r})=0,\quad [\mathbf{n}\times\mathbf{E}_n''(\mathbf{r})]_S=\mathbf{0}\tag{11}$$

where $\omega''_n > 0$, $n = 1, 2, \ldots$ are the real numbers belonging to another set of the eigenvalues of $\mathfrak{R}$.

When $\omega_0 = 0$ in (8) and hence, in (9), problem (9) has two infinite sets of *uncoupled* eigensolutions as

$$\boxed{\boxed{\mathfrak{G}_{\mathbf{E}} \{\mathbf{E}_\alpha\} :} \quad \left\{ \; \mathbf{E}_\alpha(\mathbf{r}) = \nabla \phi_\alpha(\mathbf{r}) : \quad \nabla^2 \phi_\alpha(\mathbf{r}) + \kappa_\alpha^2 \phi_\alpha(\mathbf{r}) = 0, \quad \phi_\alpha(\mathbf{r})|_S = 0 \; \right\}_{\alpha=1}^{\infty}} \tag{12}$$

where $\kappa_\alpha^2 > 0$, $\alpha = 1, 2, \ldots$ are the eigenvalues and ϕ_α are the eigenfunctions corresponding to κ_α^2's, and

$$\boxed{\boxed{\mathfrak{G}_{\mathbf{H}} \{\mathbf{H}_\alpha\} :} \quad \left\{ \; \mathbf{H}_\beta(\mathbf{r}) = \nabla \psi_\beta(\mathbf{r}) : \quad \nabla^2 \psi_\beta(\mathbf{r}) + \nu_\beta^2 \psi_\beta(\mathbf{r}) = 0, \quad \mathbf{n} \cdot \nabla \psi_\beta(\mathbf{r})|_S = 0 \; \right\}_{\beta=1}^{\infty}} \tag{13}$$

where $\nu_\beta^2 > 0$, $\beta = 1, 2, \ldots$, are the eigenvalues and ψ_β are appropriate eigensolutions. Mathematically, this means that the eigenvalue ω_0 of the operator $\mathfrak{R}$ has *infinite* power of degeneration. The sets $\{\mathbf{E}_\alpha\}_{\alpha=1}^{\infty}$ and $\{\mathbf{H}_\beta\}_{\beta=1}^{\infty}$ originate two *subspaces* in Hilbert space, $\mathfrak{G}_{\mathbf{E}}$ and $\mathfrak{G}_{\mathbf{H}}$, respectively. These subspaces involve the *irrotational* (i.e., *curl*−free) vectors only so long as $\nabla \times \mathbf{E}_\alpha = \mathbf{0}$ and $\nabla \times \mathbf{H}_\beta = \mathbf{0}$.

Denote all the solutions, $\mathfrak{X}_n$'s, to operator equation (8) as a *manifold* $\mathfrak{M}\{\mathfrak{X}_{\mathfrak{n}}\}$ and present it as follows

$$\mathfrak{M}\{\mathfrak{X}_n\} = \begin{pmatrix} \mathfrak{E}\{\mathbf{E}_{n,\alpha}\} \\ \mathfrak{H}\{\mathbf{H}_{n,\beta}\} \end{pmatrix} = \begin{pmatrix} \mathfrak{E}'\{\mathbf{E}'_n\} \oplus \mathfrak{E}''\{\mathbf{E}''_n\} \oplus \mathfrak{G}_{\mathbf{E}}\{\mathbf{E}_\alpha\} \\ \mathfrak{H}'\{\mathbf{H}'_n\} \oplus \mathfrak{H}'\{\mathbf{H}''_n\} \oplus \mathfrak{G}_{\mathbf{H}}\{\mathbf{H}_\beta\} \end{pmatrix} \tag{14}$$

where notation $\oplus$ symbolizes *direct summation* of the subspaces in the space of solutions.

Three important remarks: 1. Completeness of the manifold $\mathfrak{M}\{\mathfrak{X}_n\}$ in Hilbert space $L_2(V)$ was proved in [1] bearing on Weyl theorem from functional analysis [5]. **2.** All the six-component vectors, $\mathfrak{X}_n$'s, are mutually orthogonal in $L_2(V)$ as the eigenvectors of the self-adjoint operator $\mathfrak{R}$. Their three-component parts, $\mathbf{E}_{n,\alpha}$ and $\mathbf{H}_{n,\beta}$, are mutually orthogonal, as well. Appropriate normalizations can provide all the vectors $\mathbf{E}_{n,\alpha}$ and $\mathbf{H}_{n,\beta}$ with physical dimensions $\mathrm{V\,m^{-1}}$ (*volt per meter*) and $\mathrm{A\,m^{-1}}$ (*ampere per meter*), respectively. In this sense, manifold (14) is a *modal basis* for the cavity fields, physically. **3.** In the process of derivation of the modal basis, the time derivative, ∂_t, was saved in Maxwell's equations (2). Hence, this facilitates derivation of a problem for the modal amplitudes.

2.3 Modal Expansions of the Time-dependent Electromagnetic Quantities

The time-domain electric and magnetic field vectors are presentable via the modal expansions as

$$\begin{aligned} \mathcal{E}(\mathbf{r},t) &= \textstyle\sum_{n=1}^{\infty} e'_n(t)\,\mathbf{E}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} e''_n(t)\,\mathbf{E}''_n(\mathbf{r}) + \sum_{n=1}^{\infty} a_\alpha(t)\,\mathbf{E}_\alpha(\mathbf{r}) \\ \mathcal{H}(\mathbf{r},t) &= \textstyle\sum_{n=1}^{\infty} h'_n(t)\,\mathbf{H}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} h''_n(t)\,\mathbf{H}''_n(\mathbf{r}) + \sum_{n=1}^{\infty} b_\beta(t)\,\mathbf{H}_\beta(\mathbf{r}) \end{aligned} \tag{15}$$

where the vector elements of the basis can be considered as already known quantities coupled with their physical dimensions. The problem is to find out the scalar time-dependent coefficients which are dimensionless modal amplitudes, physically. The given functions of impressed sources, i.e., $\mathcal{J}_e$ and $\mathcal{J}_h$, from Maxwell's equations (2) should be specified appropriately and presented then via similar modal expansions as

$$\begin{aligned} \mathcal{J}_e(\mathbf{r},t) &\stackrel{def}{=} \epsilon_0\,\varpi_e\,\mathbf{e}(\mathbf{r})\;E(t) = \textstyle\sum_{n=1}^{\infty} j'_n(t)\,\mathbf{E}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} j''_n(t)\,\mathbf{E}''_n(\mathbf{r}) + \sum_{n=1}^{\infty} j_\alpha(t)\,\mathbf{E}_\alpha(\mathbf{r}) \\ \mathcal{J}_h(\mathbf{r},t) &\stackrel{def}{=} \mu_0 \varpi_h\,\mathbf{h}(\mathbf{r})\;H(t) = \textstyle\sum_{n=1}^{\infty} i'_n(t)\,\mathbf{H}'_n(\mathbf{r}) + \sum_{n=1}^{\infty} i''_n(t)\,\mathbf{H}''_n(\mathbf{r}) + \sum_{n=1}^{\infty} i_\beta(t)\,\mathbf{H}_\beta(\mathbf{r})\,. \end{aligned} \tag{16}$$

Under definition, the given source vector functions, $\mathcal{J}_e$ and $\mathcal{J}_h$, have physical dimensions of densities of the electric and magnetic currents, i.e., $\mathrm{A\,m^{-2}}$ and $\mathrm{V\,m^{-2}}$, respectively. Hence, the scalar *free parameters,* ϖ_e and ϖ_h, should have the dimension of frequency, $\mathrm{s^{-1}}$ (*inverse second*), and given vector functions of the signal carriers, $\mathbf{e}(\mathbf{r})$ and $\mathbf{h}(\mathbf{r})$, should have dimensions $\mathrm{V\,m^{-1}}$ and $\mathrm{A\,m^{-1}}$, respectively. The scalar functions $E(t)$ and $H(t)$, specifying the given time dependance of the applied signals, should be dimensionless. In the modal source expansions (16), all the time-dependent scalar coefficients are dimensionless.

2.4 Evolutionary Equations for the Modal Amplitudes: Cauchy Problems

The modal amplitude problem can be set via projecting the Maxwell's equations (with retained time derivative!) onto all the elements of modal basis. In particular, projecting the Maxwell's equations onto two coupled elements, $\mathbf{E}'_n$ and $\mathbf{H}'_n$, taken in pairs from the subspaces $\mathfrak{E}'$ and $\mathfrak{H}'$, respectively, yields a pair of ordinary differential equations with time derivative[1] for the pair of amplitudes, $e'_n(t)$ and $h'_n(t)$, as

$$\tfrac{d}{dt}e'_n+2\gamma e'_n-\omega'_n h'_n=-\varpi_e A'_n E(t)\,,\quad \tfrac{d}{dt}h'_n+\omega'_n e'_n=-\varpi_h B'_n H(t)\,;\qquad e'_n(0)=0,\quad h'_n(0)=0 \tag{17}$$

where $n=1,2,\ldots$, $\gamma=\sigma/(2\epsilon_0)$, A'_n is a projection of the vector $\mathbf{e}(\mathbf{r})$ on the element $\mathbf{E}'_n$, and B'_n is a projection of the vector $\mathbf{h}(\mathbf{r})$ on the element $\mathbf{H}'_n$. Maxwell's equations with time derivative should be supplemented with the initial conditions for the field vectors. We take them as $\mathcal{E}(\mathbf{r},0)=\mathbf{0}$ and $\mathcal{H}(\mathbf{r},0)=\mathbf{0}$ which results in the pair of initial conditions for the modal amplitudes in (17). The evolutionary equations taken jointly with the initial conditions in (17) originate a *Cauchy problem* having a single solution[2] for the modal amplitudes. Similar procedure yields one more Cauchy problem for the coupled pair of amplitudes, $e''_n(t)$ and $h''_n(t)$, as

$$\tfrac{d}{dt}e''_n+2\gamma e''_n-\omega''_n h''_n=-\varpi_e A''_n E(t)\,,\quad \tfrac{d}{dt}h''_n+\omega''_n e''_n=-\varpi_h B''_n H(t)\,;\qquad e''_n(0)=0,\quad h''_n(0)=0 \tag{18}$$

where $n=1,2,\ldots$, A''_n and B''_n are projections of the vectors $\mathbf{e}(\mathbf{r})$ and $\mathbf{h}(\mathbf{r})$ on the pair of elements, $\mathbf{E}''_n$ and $\mathbf{H}''_n$, respectively. Analogously, the Cauchy problems for the modal amplitudes of the uncoupled irrotational modes, $a_\alpha(t)$ and $b_\beta(t)$, are

$$\left\{\tfrac{d}{dt}a_\alpha+2\gamma a_\alpha=-\varpi_e A^0_\alpha\,E(t)\,,\quad a_\alpha(0)=0\right\}\quad\text{and}\quad\left\{\tfrac{d}{dt}b_\beta=-\varpi_h B^0_\beta\,H(t)\,,\quad b_\beta(0)=0\right\}. \tag{19}$$

Solving the problems (19) is elementary. Solutions of the problems (17) and (18) can be found out explicitly by the method of "matrix exponential," see [2-4]. This method need not in using of the Fourier or Laplace integral transforms, and therefore, that is available for study of various transient and regular signals.

3 Conclusion

In the presentation, a set of new graphical results will be exhibited for the modal amplitudes which appear as a result of loading a cavity by the signals describable via *1)* Dirac delta function, *2)* Heaviside step function, *3)* double-exponential function [4], *4)* sinusoidal signal having a beginning and an end in time.

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[1] Mathematician call all the differential equations, in which time derivative participates, as the *evolutionary* equations.

[2] Solution to any Cauchy problem shows how a physical system *progresses in time* (i.e., *evolves,* shortly) from its initial state (describable by given initial conditions) and up to the state at an observation time that the solution specifies just so.