On the Change in Electrostatic Potential Energy due to the Introduction of an Additional Conductor

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Abstract

This paper generalizes a result in the classical textbook by Stratton regarding the change in electrostatic potential energy due to the introduction of a conductor S_0 into a fixed system of n conductors of arbitrary shape. The change in electrostatic potential energy is rewritten as a surface integral over S_0 which connects the unperturbed problem (the electrostatic setting before S_0 is introduced) and the perturbed problem (the electrostatic setting after S_0 is introduced). The surface integral is verified by means of variable separation of Laplace's equation in bi-spherical coordinates.

1 Introduction

The work presented in this paper is motivated by the challenge of experimentally determine the electrostatic polarizability dyadic of a conducting object S_0, *i.e.*, the first moment of the induced surface charge density when S_0 is subject to a homogeneous electrostatic field of unit amplitude with sources located at infinity. The reason for this newborn interest in electrostatic quantities is because the electrostatic polarizability dyadic shows up as the fundamental quantity that restricts the all-spectrum dynamic properties of certain electromagnetic problems, see *e.g.*, [1]. The derivation in Section 2 follows the classical textbook by Stratton [2, pp. 117–118] but generalizes the results in that reference to include the effects when S_0 carries a non-zero total electric charge and when the remaining n conductors are not necessarily isolated from each other. Further developments are made in Section 3 where the volume integrals are rewritten as a surface integral over S_0 alone. This surface integral is verified in Section 4 by the means of variable separation of Laplace's equation in bi-spherical coordinates. The paper ends with some conclusions in Section 5.

2 The Change in Electrostatic Potential Energy as Integrals over V' and V_0

Consider a fixed system of n conductors S_i, where $i = 1, 2, \ldots, n$. Conductor S_i is assumed to have total electric charge Q_i and electrostatic potential Φ_i. An additional conductor S_0 with total electric charge Q_0' and electrostatic potential Φ_0' is introduced into the system. As a consequence, the total electric charge and the electrostatic potential on S_i, where $i = 1, 2, \ldots, n$, changes to Q_i' and Φ_i', respectively. Let $(\boldsymbol{E}, \boldsymbol{D})$ and $(\boldsymbol{E}', \boldsymbol{D}')$ denote the electrostatic fields in the unperturbed and perturbed problems, respectively, *i.e.*, the electrostatic settings before and after S_0 has been introduced. Then, the change in electrostatic potential energy is, by definition,

$$\varDelta W_\mathrm{e} = W_\mathrm{e}' - W_\mathrm{e} = \frac{1}{2}\iiint_{\mathbb{R}^3} \boldsymbol{E}' \cdot \boldsymbol{D}' \,\mathrm{d}v - \frac{1}{2}\iiint_{\mathbb{R}^3} \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}v. \tag{1}$$

Let V_0 denote the volume inside S_0, and let V denote the volume outside all conductors S_i, where $i = 1, 2, \ldots, n$, before S_0 has been introduced. Then $V' = V - V_0$ is the volume in which non-zero electrostatic fields exist after the introduction of S_0. Hence, (1) can be written

$$\varDelta W_\mathrm{e} = \frac{1}{2}\iiint_{V'} \boldsymbol{E}' \cdot \boldsymbol{D}' \,\mathrm{d}v - \frac{1}{2}\iiint_{V} \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}v = \frac{1}{2}\iiint_{V'} \boldsymbol{E}' \cdot \boldsymbol{D}' - \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}v - \frac{1}{2}\iiint_{V_0} \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}v. \tag{2}$$

The first integral on the right-hand side of (2) is

$$\iiint_{V'} \boldsymbol{E}' \cdot \boldsymbol{D}' - \boldsymbol{E} \cdot \boldsymbol{D} \,\mathrm{d}v = -\iiint_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D}) \,\mathrm{d}v + \iiint_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D}) \,\mathrm{d}v + \iiint_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot \boldsymbol{D}' \,\mathrm{d}v. \tag{3}$$

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We consider constitutive relations of the form $\boldsymbol{D} = \epsilon\boldsymbol{E}$ and $\boldsymbol{D}' = \epsilon\boldsymbol{E}'$ with the same proportionality factor. This gives

$$\iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v + \iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot \boldsymbol{D}'\ \mathrm{d}v = 2 \iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v. \tag{4}$$

Equation (3) thus becomes

$$\iiint\limits_{V'} \boldsymbol{E}' \cdot \boldsymbol{D}' - \boldsymbol{E} \cdot \boldsymbol{D}\ \mathrm{d}v = -\iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v + 2 \iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v. \tag{5}$$

Hence, (2) can be written

$$\mathit{\Delta} W_{\mathrm{e}} = -\frac{1}{2} \iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v + \iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v - \frac{1}{2} \iiint\limits_{V_0} \boldsymbol{E} \cdot \boldsymbol{D}\ \mathrm{d}v. \tag{6}$$

Since $\nabla \cdot (\boldsymbol{D}' - \boldsymbol{D}) = 0$ everywhere in V', we have

$$\boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D}) = -\nabla\Phi' \cdot (\boldsymbol{D}' - \boldsymbol{D}) = -\nabla \cdot (\Phi'(\boldsymbol{D}' - \boldsymbol{D})) + \Phi'\nabla \cdot (\boldsymbol{D}' - \boldsymbol{D}) = -\nabla \cdot (\Phi'(\boldsymbol{D}' - \boldsymbol{D})), \tag{7}$$

and the divergence theorem gives (the unit normal vector $\hat{\boldsymbol{n}}$ on S_i points into V')

$$\begin{aligned}\iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v = -\iiint\limits_{V'} \nabla \cdot (\Phi'(\boldsymbol{D}' - \boldsymbol{D}))\ \mathrm{d}v &= \sum_{i=0}^{n} \Phi_i' \iint\limits_{S_i} (\boldsymbol{D}' - \boldsymbol{D}) \cdot \hat{\boldsymbol{n}}\ \mathrm{d}S \\ &= \Phi_0' \iint\limits_{S_0} (\boldsymbol{D}' - \boldsymbol{D}) \cdot \hat{\boldsymbol{n}}\ \mathrm{d}S + \sum_{i=1}^{n} \Phi_i'(Q_i' - Q_i),\end{aligned} \tag{8}$$

where we have used that $\Phi' = \Phi_i'$ on S_i for $i = 1, 2, \ldots, n$. Since $\nabla \cdot \boldsymbol{D} = 0$ everywhere in V_0 and the total electric charge on S_0 is Q_0', we have

$$\iint\limits_{S_0} (\boldsymbol{D}' - \boldsymbol{D}) \cdot \hat{\boldsymbol{n}}\ \mathrm{d}S = Q_0'. \tag{9}$$

As a consequence, (8) becomes

$$\iiint\limits_{V'} \boldsymbol{E}' \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v = \Phi_0' Q_0' + \sum_{i=1}^{n} \Phi_i'(Q_i' - Q_i). \tag{10}$$

The change in electrostatic potential energy (6) can hence be written

$$\mathit{\Delta} W_{\mathrm{e}} = \Phi_0' Q_0' + \sum_{i=1}^{n} \Phi_i'(Q_i' - Q_i) - \frac{1}{2} \iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D})\ \mathrm{d}v - \frac{1}{2} \iiint\limits_{V_0} \boldsymbol{E} \cdot \boldsymbol{D}\ \mathrm{d}v. \tag{11}$$

Equation (11) generalizes the result in the classical textbook by Stratton [2, p. 118]. We get Stratton's result if we choose $Q_0' = 0$ and $Q_i' = Q_i$ for $i = 1, 2, \ldots, n$, *i.e.*, if S_0 is uncharged and the remaining n conductors are isolated from each other and from S_0.

3 The Change in Electrostatic Potential Energy as an Integral over S_0

The aim is now to rewrite the volume integrals on the right-hand side of (11) as a surface integral over S_0 alone. Since $\nabla \cdot (\boldsymbol{D}' - \boldsymbol{D}) = 0$ everywhere in V', we have

$$(\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D}) = -\nabla \cdot ((\Phi' - \Phi)(\boldsymbol{D}' - \boldsymbol{D})) + (\Phi' - \Phi)\nabla \cdot (\boldsymbol{D}' - \boldsymbol{D}) = -\nabla \cdot ((\Phi' - \Phi)(\boldsymbol{D}' - \boldsymbol{D})), \tag{12}$$

and the divergence theorem gives (the unit normal vector $\hat{\boldsymbol{n}}$ on S_i points into V')

$$\iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D}) \, \mathrm{d}v = -\iiint\limits_{V'} \nabla \cdot ((\Phi' - \Phi)(\boldsymbol{D}' - \boldsymbol{D})) \, \mathrm{d}v$$
$$= \iint\limits_{S_0} (\Phi_0' - \Phi)(\boldsymbol{D}' - \boldsymbol{D}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S + \sum_{i=1}^{n} (\Phi_i' - \Phi_i) \iint\limits_{S_i} (\boldsymbol{D}' - \boldsymbol{D}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S. \quad (13)$$

This can be simplified so that only a surface integral over S_0 remains:

$$\iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D}) \, \mathrm{d}v = \iint\limits_{S_0} (\Phi_0'\boldsymbol{D}' - \Phi_0'\boldsymbol{D} - \Phi\boldsymbol{D}' + \Phi\boldsymbol{D}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S + \sum_{i=1}^{n} (\Phi_i' - \Phi_i)(Q_i' - Q_i), \quad (14)$$

where $\Phi' = \Phi_0'$ on S_0. Since the total electric charge on S_0 is Q_0', we have

$$\iint\limits_{S_0} \Phi_0'\boldsymbol{D}' \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \Phi_0' \iint\limits_{S_0} \boldsymbol{D}' \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \Phi_0' Q_0'. \quad (15)$$

Moreover, since $\nabla \cdot \boldsymbol{D} = 0$ everywhere in V_0, the divergence theorem gives

$$\iint\limits_{S_0} \Phi_0'\boldsymbol{D} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \Phi_0' \iint\limits_{S_0} \boldsymbol{D} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \Phi_0' \iiint\limits_{V_0} \nabla \cdot \boldsymbol{D} \, \mathrm{d}v = 0 \quad (16)$$

and

$$\iint\limits_{S_0} \Phi\boldsymbol{D} \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S = \iiint\limits_{V_0} \nabla \cdot (\Phi\boldsymbol{D}) \, \mathrm{d}v = \iiint\limits_{V_0} \nabla\Phi \cdot \boldsymbol{D} \, \mathrm{d}v = -\iiint\limits_{V_0} \boldsymbol{E} \cdot \boldsymbol{D} \, \mathrm{d}v. \quad (17)$$

Hence, (14) becomes

$$\iiint\limits_{V'} (\boldsymbol{E}' - \boldsymbol{E}) \cdot (\boldsymbol{D}' - \boldsymbol{D}) \, \mathrm{d}v = \Phi_0'Q_0' + \sum_{i=1}^{n} (\Phi_i' - \Phi_i)(Q_i' - Q_i) - \iiint\limits_{V_0} \boldsymbol{E} \cdot \boldsymbol{D} \, \mathrm{d}v - \iint\limits_{S_0} \Phi\boldsymbol{D}' \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S. \quad (18)$$

Equation (11) can therefore be written

$$\mathit{\Delta} W_{\mathrm{e}} = \frac{1}{2}\Phi_0'Q_0' + \frac{1}{2}\sum_{i=1}^{n} (\Phi_i' + \Phi_i)(Q_i' - Q_i) + \frac{1}{2}\iint\limits_{S_0} \Phi\rho_S' \, \mathrm{d}S, \quad (19)$$

where we have introduced the surface charge density $\rho_S' = \boldsymbol{D}' \cdot \hat{\boldsymbol{n}}$ of the perturbed problem. Since all conductors are equipotential surfaces we can alternatively write the change in electrostatic potential energy as [3, p. 43]

$$\mathit{\Delta} W_{\mathrm{e}} = W_{\mathrm{e}}' - W_{\mathrm{e}} = \frac{1}{2}\Phi_0'Q_0' + \frac{1}{2}\sum_{i=1}^{n} \Phi_i'Q_i' - \frac{1}{2}\sum_{i=1}^{n} \Phi_iQ_i. \quad (20)$$

By comparing (19) and (20) we get a surface integral over S_0 that connects the unperturbed and perturbed problems:

$$\iint\limits_{S_0} \Phi\rho_S' \, \mathrm{d}S = \sum_{i=1}^{n} (\Phi_i'Q_i - \Phi_iQ_i'). \quad (21)$$

Note that the right-hand side of (21) is independent of Φ_0' and Q_0'. Equation (21) can also be derived by applying the divergence theorem in V' to the reciprocity-like identity $0 = -\boldsymbol{E} \cdot \boldsymbol{D}' + \boldsymbol{E}' \cdot \boldsymbol{D} = \nabla \cdot (\Phi\boldsymbol{D}' - \Phi'\boldsymbol{D})$, where we have used that $\nabla \cdot \boldsymbol{D} = \nabla \cdot \boldsymbol{D}' = 0$ everywhere in V'. The result is in agreement with (21):

$$0 = \iiint\limits_{V'} \nabla \cdot (\Phi\boldsymbol{D}' - \Phi'\boldsymbol{D}) \, \mathrm{d}v = \iint\limits_{S_0} (\Phi\boldsymbol{D}' - \Phi'\boldsymbol{D}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S + \sum_{i=1}^{n} \iint\limits_{S_i} (\Phi_i\boldsymbol{D}' - \Phi_i'\boldsymbol{D}) \cdot \hat{\boldsymbol{n}} \, \mathrm{d}S$$
$$= \iint\limits_{S_0} \Phi\rho_S' \, \mathrm{d}S + \sum_{i=1}^{n} (\Phi_iQ_i' - \Phi_i'Q_i). \quad (22)$$

Application of (21) to the change in capacitance, and its relation to the electrostatic polarizability dyadic, when S_0 is immersed into the electrostatic fields of a parallel plate capacitor, will be discussed in a forthcoming paper.

4 One conducting sphere versus two conducting spheres

Equation (21) can be verified for $n = 1$ when S_0 and S_1 are two conducting spheres by the means of bi-spherical coordinates. These curvilinear coordinates are defined by [4, pp. 1298–1301]

$$(x, y, z) = \frac{a}{\cosh \mu - \cos \eta}(\sin \eta \cos \phi, \sin \eta \sin \phi, \sinh \mu), \tag{23}$$

where $-\infty < \mu < \infty$, $0 \leq \eta \leq \pi$, and $0 \leq \phi < 2\pi$. The two spheres correspond to the coordinate surfaces $\mu = \mu_0$ and $\mu = -\mu_1$, where $\mu_0 > 0$ and $\mu_1 > 0$. If the radii of the spheres are r_0 and r_1, respectively, and their centers are separated by the distance h, then $\mu_0 = \ln((d_0 + a)/r_0)$ and $\mu_1 = \ln((d_1 + a)/r_1)$, where $a = \sqrt{d_0^2 - r_0^2} = \sqrt{d_1^2 - r_1^2}$ with $d_0 = (h^2 + r_0^2 - r_1^2)/2h$ and $d_1 = (h^2 + r_1^2 - r_0^2)/2h$. The sphere with radius r_0 is assumed to have electrostatic potential Φ_0' and total electric charge Q_0' in the perturbed problem. The analogous quantities for the sphere with radius r_1 are Φ_1' and Q_1', respectively. Laplace's equation separates in bi-spherical coordinates and the electrostatic potential in the perturbed problem can be written [4, pp. 1298–1301]

$$\Phi'(\mu, \eta) = C(\mu, \eta) \sum_{n=0}^{\infty} \left\{ \Phi_0' \frac{\mathrm{e}^{-(n+1/2)\mu}\left(\mathrm{e}^{(2n+1)(\mu+\mu_1)} - 1\right)}{\mathrm{e}^{(2n+1)(\mu_0+\mu_1)} - 1} + \Phi_1' \frac{\mathrm{e}^{(n+1/2)\mu}\left(\mathrm{e}^{(2n+1)(\mu_0-\mu)} - 1\right)}{\mathrm{e}^{(2n+1)(\mu_0+\mu_1)} - 1} \right\} \mathrm{P}_n(\cos \eta), \tag{24}$$

where $C(\mu, \eta) = \sqrt{2(\cosh \mu - \cos \eta)}$ and P_n is the Legendre polynomial of degree n. The surface charge densities on S_0 and S_1 are $\rho_{S_0}' = \epsilon_0 a^{-1}(\cosh \mu_0 - \cos \eta)\partial_\mu \Phi'(\mu_0, \eta)$ and $\rho_{S_1}' = -\epsilon_0 a^{-1}(\cosh \mu_1 - \cos \eta)\partial_\mu \Phi'(-\mu_1, \eta)$, respectively. The total electric charge on S_1 in the perturbed problem can be written

$$Q_1' = 2\pi \int_0^\pi \rho_{S_1}'(\eta) \frac{a^2 \sin \eta}{(\cosh \mu_1 - \cos \eta)^2} \, \mathrm{d}\eta. \tag{25}$$

We obtain the unperturbed problem by letting $\mu_0 \to \infty$. This implies $Q_1 = \lim_{\mu_0 \to \infty} Q_1' = 4\pi\epsilon_0 a \Phi_1 / \sinh \mu_1$, and the electrostatic potential in the unperturbed problem at the fictitious sphere $\mu = \mu_0$ becomes

$$\Phi(\mu_0, \eta) = \Phi_1 \sqrt{\frac{\cosh \mu_0 - \cos \eta}{\cosh(\mu_0 + 2\mu_1) - \cos \eta}}. \tag{26}$$

After some cumbersome algebra it is possible to verify (21) by combining the formulas above.

5 Conclusions

It is concluded that the change in electrostatic potential energy can be written either as volume integrals over V' and V_0 or as a surface integral over S_0 alone. This implies the existence of the identity (21) which connects the unperturbed and perturbed problems in a non-trivial way. This identity can be verified when $n = 1$ by solving the Laplace equation in bi-spherical coordinates, see Section 4.

References

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