Robust Semi-Analytical Integration of Singular Impedance Matrix Elements in Surface Integral Equation Solvers

Athanasios G. Polimeridis[1], *José M. Tamayo*[2], and *Juan R. Mosig*[1]

[1] Laboratory of Electromagnetics and Acoustics, Ecole Polytechnique Fédérale de Lausanne, CH-1015 Switzerland, e-mail: athanasios.polymeridis@epfl.ch; juan.mosig@epfl.ch

[2] Department of Signal Theory & Communications, Universitat Politèchnica de Catalunya, Barcelona 08034, Spain, e-mail: jose.maria.tamayo@tsc.upc.edu

Abstract

The direct evaluation method tailored to the 4-D singular integrals arising in the second-kind Fredholm surface integral equation formulations is presented. The proposed semi-analytical integration scheme extends the existing singularity cancellation methods, often used to tackle singular integrals, by utilizing a series of coordinate transformations combined with the re-ordering of the integrations. The overall algebraic manipulation results in a reduction of the dimensionality and the final 2-D kernels comprise sufficiently smooth behavior, allowing a straightforward treatment by standard quadrature rules. Numerical results demonstrate that one of the main error sources of the second-kind integral equations is undoubtedly cured.

1 Introduction

Surface integral equation (SIE) formulations find widespread use in the analysis of various electromagnetic antenna and scattering problems involving perfect electric conductors and dielectric objects. SIEs can be categorized into the Fredholm integral equations of the first and the second kind. The numerical solution of the Fredholm integral equations by means of Galerkin method of moments calls for the calculation of 4-D weakly ($1/R$) and strongly ($1/R^2$) singular integrals for the first and second kind respectively. The focal point of this work is the accurate and efficient evaluation of the following 4-D strongly singular integral:

$$I := \int_{E_P} \mathbf{g}(\mathbf{r}) \cdot \int_{E_Q} \nabla G(\mathbf{r}, \mathbf{r}') \times \mathbf{f}(\mathbf{r}') \, dS' dS \tag{1}$$

which incorporate the free-space Green's function

$$G(\mathbf{r}, \mathbf{r}') = G(R) = \frac{e^{-jkR}}{R} \tag{2}$$

with $R = |\mathbf{r} - \mathbf{r}'|$ being the distance function and $k = \omega\sqrt{\varepsilon\mu}$ the wavenumber. Here, $E_P : (\mathbf{r}_{1p}, \mathbf{r}_{2p}, \mathbf{r}_{3p}) \equiv (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ and $E_Q : (\mathbf{r}_{1q}, \mathbf{r}_{2q}, \mathbf{r}_{3q}) \equiv (\mathbf{r}_2, \mathbf{r}_1, \mathbf{r}_4)$ correspond to edge adjacent triangles lying on different planes while the discretization scheme incorporates RWG basis functions $\mathbf{f}$ and either RWG or $\hat{\mathbf{n}}\times$RWG testing functions $\mathbf{g}$ [1]. The vertex adjacent case will be presented in a forthcoming paper.

Traditionally, the above mentioned strongly singular integrals by means of the singularity cancellation [2, 3] and singularity subtraction [4–6] methods. Unfortunately, both methods fail to meet the requirements for the state-of-the-art SIE solvers, especially when the quest for machine precision is combined with the need for improved efficiency. Herein, the direct evaluation method [7, 8] specially tailored to the singular integrals under investigation is presented, thus completing the research undertaken by the authors previously and which was focusing on the weakly singular integrals [9, 10]. The fast and accurate evaluation of the Galerkin impedance matrix elements will allow a safe shift of future research studies on the other (of different nature) well-known problems of the second-kind Fredholm SIE formulations.

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2 Direct Evaluation Method

The first step of the direct evaluation method is to introduce an appropriate parameter space. In this manuscript, an equilateral parameter space $\{\eta, \xi\}$, where $-1 \leq \eta \leq 1$, $0 \leq \xi \leq \sqrt{3}(1-|\eta|)$ is employed for each one of the triangles, following the original work by Gray et al. [7, 8]:

$$\mathbf{r} = \frac{\mathbf{r}_2 + \mathbf{r}_1}{2} + \left[\frac{\mathbf{r}_2 - \mathbf{r}_1}{2}, \quad \frac{2\mathbf{r}_3 - \mathbf{r}_1 - \mathbf{r}_2}{2\sqrt{3}}\right] \begin{bmatrix} \eta \\ \xi \end{bmatrix}. \tag{3}$$

The strongly singular integral (1) in the new parametric space can be evaluated as

$$I = (J_p J_q) \int_{-1}^{1} d\eta \int_{0}^{\xi(\eta)} d\xi \int_{-1}^{1} d\eta' \int_{0}^{\xi(\eta')} \mathbf{g}(\eta, \xi) \cdot (\nabla G(R) \times \mathbf{f}(\eta', \xi'))\, d\xi' \tag{4}$$

Afterwards, we perform two consecutive polar coordinate changes:

$$\begin{array}{l} \eta' = \rho\cos(\theta) - \eta \\ \xi' = \rho\sin(\theta) \end{array} \quad \text{and} \quad \begin{array}{l} \rho = \Lambda\cos(\Psi) \\ \xi = \Lambda\sin(\Psi) \end{array} \tag{5}$$

The distance function in the new parametric system takes the form

$$R = \Lambda\sqrt{\beta_{e_1}\sin^2(\Psi) + \beta_{e_2}\sin(\Psi)\cos(\Psi) + \beta_{e_3}\cos^2(\Psi)} = \Lambda B(\theta, \Psi) \tag{6}$$

and the original integral can be written as a sum of 3-D integrals,

$$I = \sum_{l=0}^{1}\sum_{m=0}^{1} \int_{-1}^{1} d\eta \int_{\Theta_l}^{\Theta_{l+1}} d\theta \int_{\Psi_m}^{\Psi_{m+1}} A(\eta, \theta, \Psi) d\Psi \tag{7}$$

since the integral over Λ, can be evaluated analytically. Note that the final Jacobian after the whole set of parametric transformations is given by

$$J = \frac{A_p A_q}{3}\Lambda^2\cos(\Psi) \tag{8}$$

while the strongly singular kernel can be extended as

$$\nabla G(R) = -\frac{e^{-jk\Lambda B(\theta,\Psi)}}{\Lambda^2 B(\theta, \Psi)^3}\left(1 + jk\Lambda B(\theta, \Psi)\right)\mathbf{B}(\theta, \Psi). \tag{9}$$

In fact, one of the key points of the direct evaluation method is that the singularity $1/\Lambda^2$ is canceled out with the Jacobian (8). Furthermore, the result is in turn analytically integrable in terms of η. Therefore, it is interesting to suitably change the order of the integrals placing the integration with respect to η right after the integration with respect to Λ. Finally, the integral (1) takes the form:

$$\begin{aligned} I = &\int_0^{\pi/3} d\theta \int_0^{\Psi_B} \chi^a(\theta, \Psi) d\Psi + \int_{\pi/3}^{\pi/2} d\theta \int_{\Psi_A}^{\Psi_B} \chi^a(\theta, \Psi) d\Psi + \int_{\pi/3}^{\pi/2} d\theta \int_0^{\Psi_A} \chi^b(\theta, \Psi) d\Psi \\ &+ \int_{\pi/2}^{\pi} d\theta \int_0^{\Psi_A} \chi^c(\theta, \Psi) d\Psi + \int_0^{\pi/2} d\theta \int_{\Psi_B}^{\pi/2} \chi^d(\theta, \Psi) d\Psi + \int_{\pi/2}^{\pi} d\theta \int_{\Psi_A}^{\pi/2} \chi^d(\theta, \Psi) d\Psi \end{aligned} \tag{10}$$

being $\chi^\alpha(\theta, \Psi)$ with $\alpha \in \{a, b, c, d\}$ analytic 2-D sufficiently smooth functions depending on the angle parameters θ and Ψ. The above mentioned highly abbreviated derivation will be presented in a very compact and ready-to-use form at the conference presentation so that the direct evaluation method finds its place in standard mathematical subroutine libraries widely used in the computational electromagnetics community.

3 Implementation Considerations

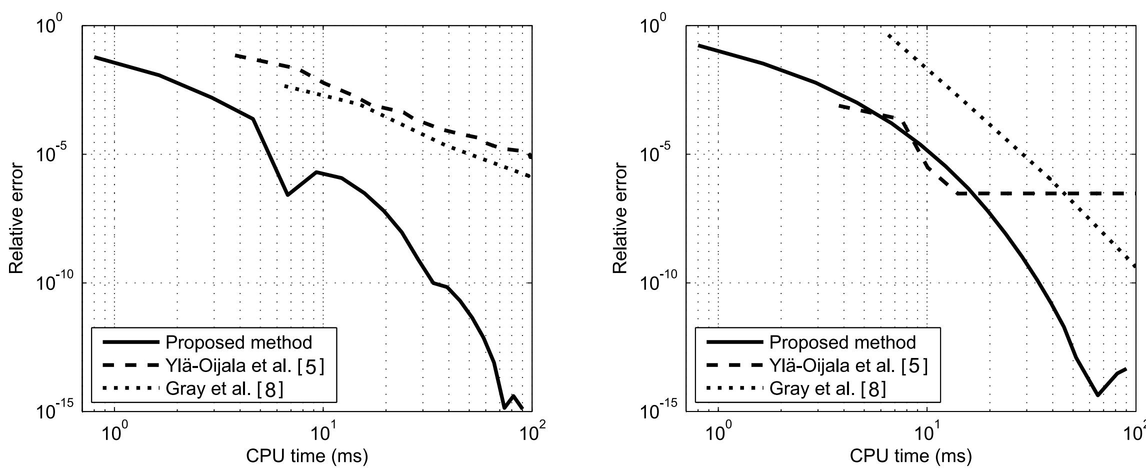
The overall analytic integrations required in the direct evaluation method can become prohibitively complicated, jeopardizing its general-purpose character. A smart way to overcome this obstacle is to utilize one of the proven symbolic mathematical software like $Maple$. In this framework, changing the basis/testing functions represents only modifying a couple of code lines. The generation of each new 2-D function for a combination of basis/testing functions takes no more than a few seconds. Moreover, it automatically creates the necessary $Matlab$ or C codes in the given file names, allowing implementation in both programming platforms. Some preliminary numerical experiments undertaken by the authors, though, showed that the later comprises a superior performance in the evaluation of the 4-D strongly singular integrals under study, as it consumes considerably less computational time.

4 Numerical Results

A study on the accuracy and efficiency of the proposed method is shown in this section. For RWG and $\hat{\mathbf{n}}\times$RWG basis/testing functions ($\mathbf{f}$ and $\mathbf{g}$), it suffices to analyze among the various combinations a single, albeit representative, case. In particular, we choose the case $\mathbf{f}(\mathbf{r}') = \mathbf{f}_1'(\mathbf{r}')$ and $\mathbf{g}(\mathbf{r}) = \mathbf{f}_3(\mathbf{r})$, where

$$\mathbf{f}_3(\mathbf{r}) = \frac{|\mathbf{r}_2 - \mathbf{r}_1|}{2A_p}(\mathbf{r} - \mathbf{r}_3); \quad \mathbf{f}_1^{'}(\mathbf{r}') = \frac{|\mathbf{r}_4 - \mathbf{r}_2|}{2A_q}(\mathbf{r}' - \mathbf{r}_2). \tag{11}$$

We consider two edge adjacent triangles (T1:$\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3\}$ and T2:$\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_4\}$) with the following vertices: $\mathbf{r}_1 = [0, 0, 0]$; $\mathbf{r}_2 = [0, 0.1\lambda, 0]$; $\mathbf{r}_3 = [0, 0, 0.1\lambda]$; $\mathbf{r}_4 = [0.1\lambda, 0, 0]$, where $\lambda = 1$ [m] corresponds to the wavelength. Note that $\mathbf{r}_1$ and $\mathbf{r}_2$ are the two common vertices. Moreover, the reference solution has been obtained by



(a) Real part. (b) Imaginary part.

Figure 1: Comparison of the relative error as a function of the CPU time for the triangles T1 and T2.

means of the proposed method in combination with the symbolic software *Maple*, using high precision arithmetic (300 digits) in all the computations together with a high number of integration points until it converges smoothly up to any desired accuracy. The exact value up to 32 significant digits equals

$$I = 3.4928883683897266018383577695620 \cdot 10^{-3} - j2.2540732129690316163209769145458 \cdot 10^{-5}. \tag{12}$$

According to the results, as depicted in Fig. 1, the regularization of the strongly singular kernel together with the reduction of the dimensionality of the original integral from 4-D to 2-D via the direct evaluation method has resulted in formulas which provide unmatched accuracy with significantly reduced computational effort. In particular, using codes implemented in MATLAB® and run in a computer with an Intel® Core 2 Quad, 2.83 GHz (no parallelization has been done), Linux 2.6.28 Ubuntu and MATLAB® 7.7.0.471, the proposed scheme reaches a relative error smaller than 10^{-5} in 10ms and almost machine precision in about less than 100ms.

5 Conclusion

In this work, one of the main error sources of the second-kind Fredholm integral equations is tackled by means of the direct evaluation method. The key feature of the proposed scheme lies on the appropriate regularization of the singular integrand together with the reduction of the dimensionality of the original integral from 4-D to 2-D. The procedure presented herein succeed in providing numerical results of unmatched accuracy (close to the machine precision) and efficiency, thus, improving substantially the accuracy of the impedance matrix elements in surface integral equation formulations as well as reducing the overall filling time.

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