Impedance Boundary Condition on a PEC Backed Uniaxially Anisotropic Sheet of Arbitrary Shape

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Abstract

Impedance boundary condition is investigated on a PEC backed uniaxially anisotropic sheet of arbitrary shape based on the distributional tools on an arbitrary regular surface. The investigation provides a generalization of the distributional procedure introduced by İdemen in 1993 for the case when the material sheet is of arbitrary shape and backed by a perfect electrical conductor. The availability of an impedance boundary condition is scrutinized for 8 different types of materials of practical interest.

1. Introduction

Simulation of natural or man-made thin layers by approximate boundary conditions is one of the most important directions of research in scattering theory connected with complementing the associated boundary value problems in antenna engineering, radio wave propagation and planar microwave technologies. The historical development and descriptions of a large variety of such conditions including resistive, conductive, standard impedance, generalized impedance, and absorbing boundary conditions as well as a systematic treatment of scattering by canonical bodies simulated by such conditions has been introduced in 1995 in the monograph [1].

In the present work we provide the end results for the availability of an impedance boundary condition on a closed, arbitrarily shaped and PEC backed thin material surface simulated by uniaxially anisotropic constitutive parameters based on the distributional tools on an arbitrary regular surface (cf.[2]) and the methodology devised by İdemen in [3] for uniaxially anisotropic planar sheets. The problem of radiation and scattering by anisotropic thin sheets has earned much importance in parallel to their use in microwave technology as substrate layers. One may refer to [4] and [5] for a historical review and early developments in that field. In that regard the present work is focused only on analytical aspects with no specific area of application addressed.

The methodology introduced in [3] and extended in the present work in simulating thin material surfaces has the following features:

i) The analytical tools are adopted from the powerful theory of Schwartz - Sobolev distributions and therefore the methodology is capable of taking into account polarization mechanisms of all orders inside the material sheet and yield elegant results in a straightforward manner through general theorems on the orders of field singularities.

ii) The derived boundary relations apply regardless of the structures, frequency, polarizations and locations of the sources outside the material sheet as well as the constitutive parameters of the medium surrounding the sheet. Without losing generality, in the present work we assume the ambient medium has simple constitutive parameters (see Fig.1).

iii) The fields inside the material sheet are approximated by a zeroth order averaging procedure which relates the fields inside and on the two interfaces of the thin sheet - thereby yielding first order boundary relations.

iv) The methodology can also be extended to include the results for biaxially and general anisotropic as well as multilayered sheets with the motivation of investigating scattering by integrated planar microwave circuit structures.

Due to space limitation we are only able to provide the end results while we also conform utmost to the notation introduced in [3]. We carry out the investigation for phasor quantities omitting time dependence $e^{-i\omega t}$.

2. Surficial Vector Differential Operators

Let (u_1, u_2) be the real valued parametric curves of a two-sided, regular surface S described by the position vector $\vec{r}_S = \vec{r}(u_1, u_2)$. A quantity which assumes one or more definite values at each point of a surface is called a function of position, or a "point function" for the surface. Let us consider a scalar and a vector point function

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$$\psi(u_1,u_2)\ ,\ \ \vec{A}(u_1,u_2)=A_1(u_1,u_2)\hat{u}_1+A_2(u_1,u_2)\hat{u}_2+A_n(u_1,u_2)\hat{n}$$

where $\hat{u}_1$, $\hat{u}_2$ are unit tangent vectors along the curves $u_1=const.$ and $u_2=const.$ and $\hat{n}$ is the unit normal of S, which constitute a right handed system. Then it can be shown [6,Ch.12] that the gradient and divergence operators acting on the point functions on a surface are given as

$$grad_S\psi=\frac{\hat{u}_1}{h_1}\frac{\partial\psi}{\partial u_1}+\frac{\hat{u}_2}{h_2}\frac{\partial\psi}{\partial u_2}\ ,\ div_S\vec{A}=\frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2A_1)+\frac{\partial}{\partial u_2}(h_1A_2)\right]-2\Omega A_n \tag{1a,b}$$

Here h_1, h_2 are the metric coefficients of the parametric curves; the principle radii of curvature $\alpha_{1,2}$ are related to the metric coefficients through $1/\alpha_1=-(1/h_1)dh_1/dn$, $1/\alpha_2=-(1/h_2)dh_2/dn$ and $2\Omega=1/\alpha_1+1/\alpha_2=-div_S(\hat{n})$ is called the first curvature of S.

3. Simulation of a PEC Backed Uniaxially Anisotropic Sheet

Let the uniaxially anisotropic constitutive tensors of the sheet be given in matrix form as

$$\bar{\bar{\varepsilon}}=\begin{bmatrix}\varepsilon & 0 & 0\\ 0 & \varepsilon & 0\\ 0 & 0 & \varepsilon_n\end{bmatrix}\ ,\ \bar{\bar{\mu}}=\begin{bmatrix}\mu & 0 & 0\\ 0 & \mu & 0\\ 0 & 0 & \mu_n\end{bmatrix}\ ,\ \bar{\bar{\sigma}}=\begin{bmatrix}\sigma & 0 & 0\\ 0 & \sigma & 0\\ 0 & 0 & \sigma_n\end{bmatrix} \tag{2a-c}$$

while the ambient region II has constant electrical parameters ($\varepsilon_2,\mu_2,\sigma_2$) as illustrated in Fig. 1. The thin sheet is assumed to have a thickness $2d$ and the fields inside the PEC are set to zero.

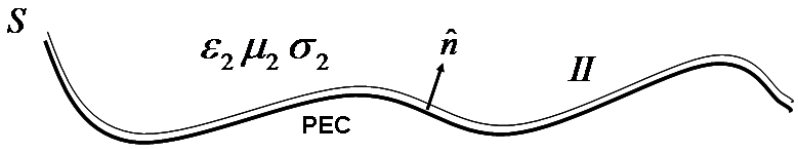

Fig. 1. A cross section of an arbitrarily shaped PEC backed material sheet in a simple medium

The corresponding complex permittivities shall be given as

$$\varepsilon_2^c=\varepsilon_2+i\sigma_2/\omega\ \ ,\ \ \varepsilon^c=\varepsilon+i\sigma/\omega\ ,\ \ \varepsilon_n^c=\varepsilon_n+i\sigma_n/\omega \tag{3a-c}$$

where we define

$$e=-i\omega d(\varepsilon^c-\varepsilon_0)\ \ ,\ \ h=-i\omega d(\mu-\mu_0) \tag{4a,b}$$

$$e_{2n}=-d\frac{\varepsilon_2^c}{\varepsilon_0}\frac{(\varepsilon_n^c-\varepsilon_0)}{\varepsilon_n^c}\ \ ,\ \ h_{2n}=-d\frac{\mu_2}{\mu_0}\frac{(\mu_n-\mu_0)}{\mu_n}\ . \tag{4c,d}$$

The jump relations derived for the 8 types of materials in Table 1 are provided in Table 2 with comments on the availability and form of the impedance boundary condition. It is observed that impedance type boundary conditions are available only for 4 (out of 8) types of materials in Table 1. Regarding the notation, the field quantities in the form $\vec{A}^{II}$ denote the limiting values of a vector $\vec{A}$ as one approaches on a point on S from the region II, while the subscripts t and n indicate the components of any field quantity tangential and normal to S which satisfy $\vec{A}=\vec{A}_t+\hat{n}A_n=\left(\hat{n}\times\vec{A}\right)\times\hat{n}+\hat{n}\left(\hat{n}\cdot\vec{A}\right)$. $\bar{\bar{I}}$ is the unit tensor, $\vec{J}_S$ represents the current (density) flowing on the PEC backing and $\vec{J}_C^{II}$ is the conduction current (density) in region *II*. As a general comment covering all 8 types of materials one can say that an impedance boundary condition can be obtained (regardless of the operating frequency) as long as the following three conditions are simultaneously satisfied:

i) $\mu_n=\mu_0$

ii) $\varepsilon\neq\varepsilon_0$ or $\sigma\neq 0$

iii) $\varepsilon_n\neq\varepsilon_0$ or $\sigma_n\neq 0$ or $\mu\neq\mu_0$

4. Conclusion

We observe that the analytical results obtained for an arbitrarily shaped material sheet has the same formal representation as for a planar one derived in [3], while the first curvature parameter 2Ω is observable only in the distributional forms of the Gauss Laws (third and fourth equations) as embedded inside the surficial divergence operator div_S .

Table 1. Physical parameters of certain types of uniaxially anisotropic materials

	TYPE	SPECIFIC PARAMETERS	CONSTITUTIVE PARAMETERS
1	GENERAL UNIAXIALLY ANISOTROPIC SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_n \end{bmatrix}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
2	UNIAXIALLY ANISOTROPIC DIELECTRIC SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_n \end{bmatrix}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
3	UNIAXIALLY ANISOTROPIC CONDUCTOR SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma_n \end{bmatrix}$
4	UNIAXIALLY ANISOTROPIC MAGNETIC SHEET	$e = e_{1n} = e_{2n} = 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_n \end{bmatrix}$ $\bar{\bar{\sigma}} = 0$
5	RESISTIVE SHEET	$e_{1n} = e_{2n} = 0$ $h = h_{1n} = h_{2n} = 0$ $R = \frac{1}{2e} = \frac{i}{2\omega d[\varepsilon - \varepsilon_0 + i\sigma/\omega]}$	$\bar{\bar{\varepsilon}} = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon_0 \end{bmatrix}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{bmatrix}$
6	CONDUCTIVE SHEET	$e = e_{1n} = e_{2n} = 0$ $h_{1n} = h_{2n} = 0$ $G = -\frac{1}{2h} = \frac{-i}{2\omega d(\mu - \mu_0)}$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu_0 \end{bmatrix}$ $\bar{\bar{\sigma}} = 0$
7	ISOTROPIC LOSSY DIELECTRIC SHEET	$e, e_{1n}, e_{2n} \neq 0$ $h = h_{1n} = h_{2n} = 0$	$\bar{\bar{\varepsilon}} = \varepsilon \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu_0 \bar{\bar{I}}$ $\bar{\bar{\sigma}} = \sigma \bar{\bar{I}}$
8	ISOTROPIC MAGNETIC SHEET	$e = e_{1n} = e_{2n} = 0$ $h, h_{1n}, h_{2n} \neq 0$	$\bar{\bar{\varepsilon}} = \varepsilon_0 \bar{\bar{I}}$ $\bar{\bar{\mu}} = \mu \bar{\bar{I}}$ $\bar{\bar{\sigma}} = 0$

Table 2. Boundary relations on certain types of PEC backed uniaxially anisotropic material sheets of arbitrary shape

TYPE	BOUNDARY RELATIONS
1	$\hat{n} \times \vec{E}_t^{II} = \hat{n} \times \left[(e_n / i\omega)\, grad_S div_S \vec{J}_S + e_{2n}\, grad_S E_n^{II} \right] - h \left[\vec{J}_S \times \hat{n} + \vec{H}_t^{II} \right]$ $\hat{n} \times \vec{H}_t^{II} = h_{2n} \hat{n} \times grad_S H_n^{II} + e \vec{E}_t^{II}$ $i\omega D_n^{II} = J_{Cn}^{II} + e\, div_S \vec{E}_t^{II}$ or $i\omega \varepsilon_2^c E_n^{II} = e\, div_S \vec{E}_t^{II}$ $i\omega B_n^{II} = h\, div_S \left[\vec{J}_S \times \hat{n} + \vec{H}_t^{II} \right]$ *Comment:* The impedance boundary condition $\vec{E}_t^{II} = Z\, \hat{n} \times \vec{H}_t^{II}$ appears in the second equation for $h_{2n} = 0$ ($\mu_n = \mu_0$) with $Z = 1/e = i \Big/ \left[\omega d (\varepsilon^c - \varepsilon_0) \right]$ regardless of the rest of the constitutive parameters.

2,3,7	$\hat{n}\times\vec{E}_t^{II} = \hat{n}\times\left[(e_n/i\omega)\, grad_S div_S \vec{J}_S + e_{2n}\, grad_S E_n^{II}\right]$ $\hat{n}\times\vec{H}_t^{II} = e\vec{E}_t^{II}$ $i\omega D_n^{II} = J_{Cn}^{II} + e\, div_S \vec{E}_t^{II}$ or $i\omega\varepsilon_2^c E_n^{II} = e\, div_S \vec{E}_t^{II}$ $i\omega B_n^{II} = 0$, i.e. $B_n^{II} = 0$, $H_n^{II} = 0$ *Comment:* The impedance boundary condition appears in the second equation naturally with $Z = 1/e = i\Big/\left[\omega d(\varepsilon^c - \varepsilon_0)\right]$. For a Type 3 material the impedance becomes real valued as $Z = 1/\sigma d$.
4,8	$\hat{n}\times\vec{E}_t^{II} = -h\left[\vec{J}_S\times\hat{n} + \vec{H}_t^{II}\right]$ or $\vec{E}_t^{II} = h\left[\vec{J}_S + \hat{n}\times\vec{H}_t^{II}\right]$ $\hat{n}\times\vec{H}_t^{II} = h_{2n}\hat{n}\times grad_S H_n^{II}$ $i\omega\varepsilon_2^c E_n^{II} = 0$, i.e. $E_n^{II} = 0$ $i\omega B_n^{II} = h\, div_S\left[\vec{J}_S\times\hat{n} + \vec{H}_t^{II}\right] = h\, div_S\left(\hat{n}\times\vec{E}_t^{II}\right)$ *Comment:* We cannot obtain an impedance boundary condition.
5	$\hat{n}\times\vec{E}_t^{II} = \vec{0}$, i.e. $\vec{E}_t^{II} = \vec{0}$ $\hat{n}\times\vec{H}_t^{II} = e\vec{E}_t^{II}$ so that $\hat{n}\times\vec{H}_t^{II} = \vec{0}$, i.e. $\vec{H}_t^{II} = \vec{0}$ $i\omega D_n^{II} = J_{Cn}^{II} + e\, div_S \vec{E}_t^{II}$ or $i\omega\varepsilon_2^c E_n^{II} = e\, div_S \vec{E}_t^{II} = 0$, i.e. $E_n^{II} = 0$ $i\omega B_n^{II} = h\, div_S\left[\vec{J}_S\times\hat{n} + \vec{H}_t^{II}\right] = 0$,i.e. $B_n^{II} = 0$, $H_n^{II} = 0$ *Comment:* We obtain the end results $\vec{E}^{II} = \vec{0}$, $\vec{H}^{II} = \vec{0}$, which reveal that the thin sheet simulation breaks down for this parametrization.
6	$\hat{n}\times\vec{E}_t^{II} = -h\left[\vec{J}_S\times\hat{n} + \vec{H}_t^{II}\right] = -h\vec{J}_S\times\hat{n}$, i.e. $\vec{E}_t^{II} = h\vec{J}_S$ $\hat{n}\times\vec{H}_t^{II} = \vec{0}$, i.e. $\vec{H}_t^{II} = \vec{0}$ $i\omega D_n^{II} = J_{Cn}^{II}$ or $i\omega\varepsilon_2^c E_n^{II} = 0$, i.e. $E_n^{II} = 0$ $i\omega B_n^{II} = h\, div_S\left[\vec{J}_S\times\hat{n} + \vec{H}_t^{II}\right] = h\, div_S\left(\vec{J}_S\times\hat{n}\right) = div_S\left(\vec{E}_t^{II}\times\hat{n}\right)$ *Comment:* We cannot obtain an impedance boundary condition.

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