A Lagrange Polynomial Chebyshev Pseudo Spectral Time Domain Method in One Dimensional Large Scale Applications

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1. Abstract

Pseudo Spectral Time Domain method based on Discrete Fourier series has been widely used in computational electromagnetics. However, this method has some disadvantages such as, the Gibbs phenomena, source conditioning and errors due to interpolation and staircase modeling of complex objects. To overcome these limitations, a Lagrange Polynomial Chebyshev Pseudo Spectral Time Domain method has been proposed. In this work, the efficiency of this method for large scale problems is examined in the sense of numerical dispersion errors (accuracy) by solving one dimensional wave equation in a simple medium. The numerical results are compared for validation with the analytical solution and standard Finite Difference Time Domain method solution.

2. Introduction

Spectral Time Domain methods are a generalization of separation of variable techniques in the time domain. They are favorable when direct analytical time domain solution is desired [1]. However, because practical electromagnetic problems consist of complex geometries, a numerical version of Spectral Time Domain method is needed. Pseudo Spectral Time Domain (PSTD) methods are proposed for this necessity [2, 3]. Fundamental idea in the PSTD methods is to use an interpolation function for global calculation of spatial derivatives at discrete (collocation) points which are also known as Pseudospectral approximation. The most important feature of the PSTD methods is the high (infinite) order of accuracy when calculating the spatial derivatives. This makes the PSTD method superior over the standard Finite Difference Time Domain (FDTD) method. This is especially important in the large scale time domain electromagnetic problems due to the cumulative behavior of the numerical dispersion errors. Furthermore the convergence of the PSTD methods is faster than the standard FDTD method [2].

Two fundamental steps in the PSTD method are to choose type of interpolation function and collocation points. Although different versions of the PSTD methods are present in the literature of different scientific disciplines, Fourier Pseudo Spectral Time Domain (F-PSTD) method is widely used in numerical electromagnetic problems [4]. The F-PSTD method uses trigonometric functions as the interpolation function with uniform (equidistance) discrete points used as the collocation points. An advanced version of it uses non-uniform collocation points with transformation to uniform one. However, both of the F-PSTD methods has the following disadvantages:

- Material discontinuities with sharp gradients (such as high conductance) causes Gibbs phenomena,
- Source induction must be handled with a special care in order to avoid Gibbs phenomena,
- Errors comes from interpolation (in the sense of Lebesque constant) and staircase discretization of the objects,
- Absorbing boundary condition has to be implemented for non-periodic problems.

In order to overcome these difficulties, a Lagrange Polynomial Chebyshev Pseudo Spectral Time Domain method is considered in this work [5, 6]. The PSTD methods using the Chebyshev collocation points are called Chebyshev PSTD (CPSTD) methods. If the method also uses Lagrange polynomials as interpolation function, then the method is named further, Lagrange Polynomial CPSTD (L-CPSTD) method. It is known that Chebyshev nodes give minimum interpolation errors in the sense of Lebesque constant. One of the fundamental weaknesses of the L-CPSTD method is unit time step restriction for numerical stability. The order of this restriction is $O(N^{-2})$ where N is number of collocation points [7, 8], but it is $O(N^{-1})$ in standard FDTD method. The mapping techniques are applied to overcome this restriction with better accuracy of the L-CPSTD method [9]. Furthermore, the multi-domain approach of the L-CPSTD method for complex geometries can also be used in the content of domain decomposition techniques [10].

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3. Lagrange Polynomial Pseudo Spectral Time Domain Method

The L-CPSTD method uses Lagrange polynomials as the interpolation function and Chebyshev nodes as the collocation points. In order to show accuracy and effectiveness of the L-CPSTD method for the solution of the large scale problems, one dimensional wave equation is considered in the simple medium

$$\frac{\partial^2}{\partial y^2}u(y,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}u(y,t) = 0 \tag{1}$$

where c is the speed of light. The field function $u(y,t)$ is spectrally decomposed into basis functions, at the discrete points of y_j , $(j = 0,1..N)$ as

$$u(y_j,t) = \sum_{n=0}^{N} a_n(t)\psi_n(y_j) \tag{2}$$

where $a_n(t)$ is time dependent coefficient, $\psi_n(y_j)$ is interpolation function evaluated at the discrete points and N is total number of collocation points for approximation. When Lagrange polynomials are chosen as the interpolation function, then the spatial derivation of field function at the collocation points can be interpreted as a matrix form

$$\frac{\partial^2}{\partial y^2}u(y_j,t) \cong D_N^2 u(y_j,t) \tag{3}$$

where $D_N^2 = D_N \times D_N$ is the second order differentiation matrix of Lagrange interpolation polynomials. D_N can be formulated as an explicit form with the proper coefficients [11]. The calculation of D_N can also be performed by using Fast Fourier Transformation with less computation [12].

Using the central finite difference formula for second order time derivation and the formula for the Lagrange polynomial based second order spatial derivation, the update equation of the L-CPSTD method is obtained as

$$u^{n+1} = 2u^n - u^{n-1} + (c\Delta t)^2 D_N^2 u^n \tag{4}$$

where Δt is unit time step and the notation in the update equations for discrete field values is used as $u^n = u(j, n\Delta t)$. Because the L-CPSTD method is global, the spatial calculation points of the field values in the update equation cover all collocation points in every time step. In order to excite the fields, a soft or hard type source function with chosen (arbitrary) time dependency must be applied as an initial condition.

The choice on the distribution of collocation (grid) points is also critical in the PSTD methods. There are two main options as uniform and non-uniform points. The L-CPSTD method uses the Chebyshev-Lobatto collocation points which are the zeros of the Chebyshev polynomials, defined as

$$y_j = cos\left(j\frac{\pi}{N}\right) \tag{4}$$

The reason of using of the Chebyshev-Lobatto collocation points (non-uniform) is that they minimize the interpolation errors in the sense of Lebesgue constant [12]. The temporal derivatives can also be approximated by different implicit and semi-implicit techniques. The choice of these techniques affects the accuracy but its major impact is seen on the stability of the L-CPSTD method. The stability analysis of the L-CPSTD method is not well understood as in the standard FDTD method. The eigenvalue analysis of the differentiation matrix must be handled for the stability analysis of the L-CPSTD method. The pseudospectra of the differentiation matrix instead of its spectra have to be considered in the stability analysis due to the non-normal behavior of the L-CPSTD differentiation matrix. The effect of the boundary conditions is also taken into consideration for the stability of the L-CPSTD method. Moreover it is known that the differentiation matrix of the L-CPSTD method for the second order spatial derivative is more robust to the numerical deficiencies rather than the first order one. This is because; the eigenvalues of the second order spatial differentiation matrix is more close to the physical models.

4. Numerical Result

In order to show the accuracy and efficiency of the L-CPSTD method for large scale problems, one dimensional wave equation in the simple medium is solved. The considered simulation setup as an example of the large scale problems is shown in Figure 1. The problem space lies from the discrete point y_0 to discrete point y_N. The total length of the problem space is 200λ for evaluating the boundary interactions. The excited field at y_0 is observed at y_{obs}. The distance between the source and observation point is $L_y = 100\lambda$. A hard point source by a first-derivative of Blackman-Harris window with a center frequency of $25\ MHz$ is applied into the solution as an initial condition. In Figure 2, the time dependency of the field at 100λ away from the source is shown with comparison of the results from the analytical and the L-CPSTD and the standard FDTD method solution. Specially two different FDTD results for $\lambda/20$ and $\lambda/100$ unit spatial discretization is given. The superiority of the L-CPSTD method is clear in the sense of accuracy with the π points per wavelength (λ/π) in average ($\lambda/4$ or $\lambda/5$ in practice). The difference between the numerical results of the L-CPSTD and the standard FDTD method is not easily distinguishable. However, the standard FDTD solution uses the 100 points per wavelength ($\lambda/100$) in order to reach the close level of accuracy, but with not negligible phase difference. Although the unit time step of the L-CPSTD method is smaller than the standard FDTD method, it is possible to overcome this problem by mapping which makes the L-CPSTD method also more accurate [12]. Specially, the unit time step of the standard FDTD method is taken as equal to the time step of the L-CPSTD method in order to alleviate errors comes from different values of the time steps.

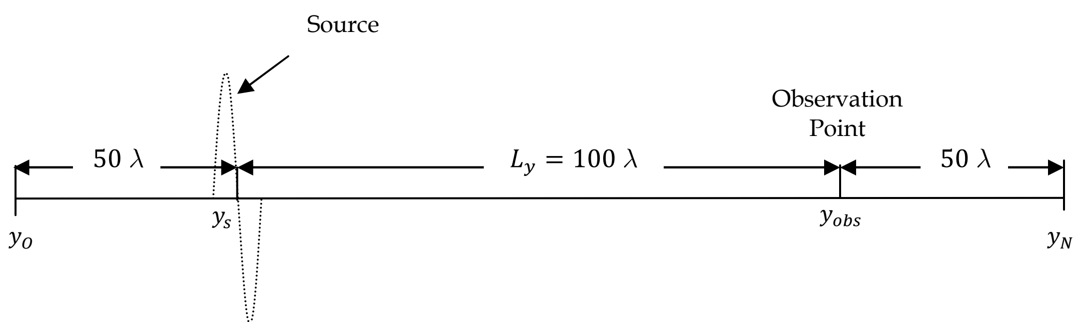

Figure 1. Considered simulation setup.

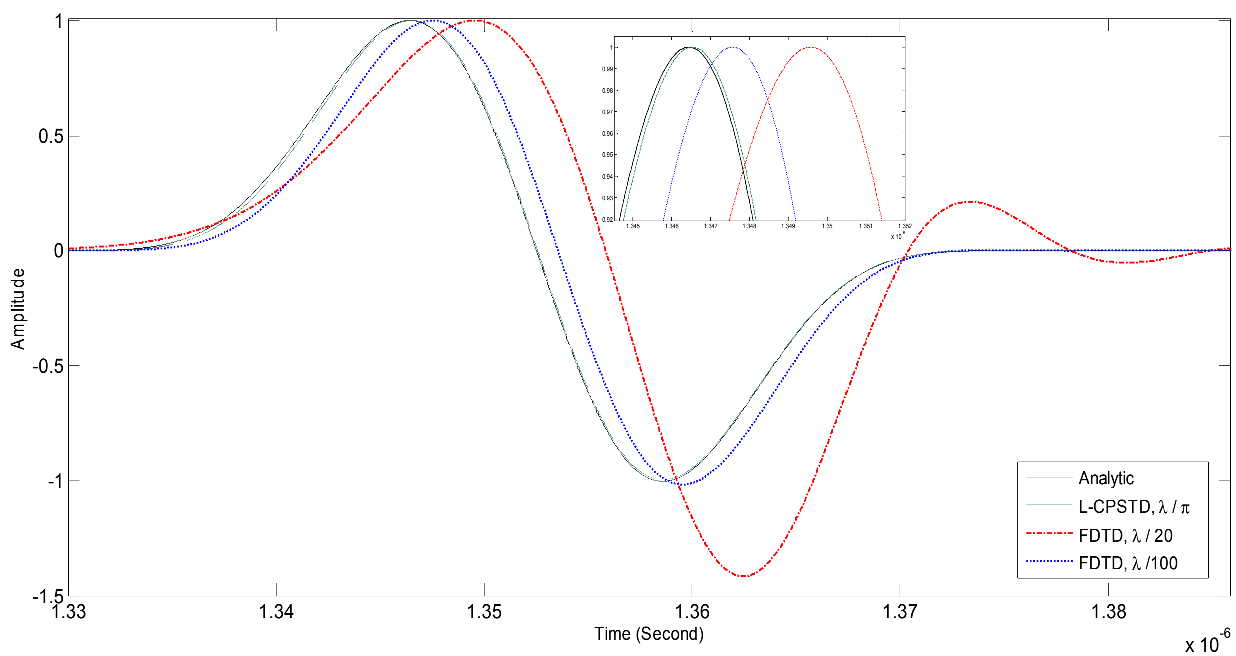

Figure 2. Comparison of time domain signals (propagated 100λ away from the source) between the analytical, the standard FDTD and the L-CPSTD methods.

5. Conclusion

The solution of the one dimensional wave equation in the simple medium is given by the L-CPSTD method in this work. The accuracy in the content of the numerical dispersion error for the L-CPSTD method is shown for especially in the case of the large scale problem. The numerical results obtained with the L-CPSTD method are compared with the result of the analytical and the standard FDTD method solutions. The results indicate that the L-CPSTD methods for large scale problem perform better when dispersion errors are the primary concern. For the future works, two and three dimensional problems for the wave and Maxwell equations with absorbing boundary conditions are being developed for solving scattering or other open boundary problems. Mapping and multi-domain approaches of the L-CPSTD method are also planned to apply for scattering problems of the complex geometries.

6. References

[1] S. Aksoy, and O. A. Tretyakov, "The evolution equations in study of the cavity oscillations excited by a digital signal," *IEEE Trans. on Antennas and Propagation*, vol. 52, no. 1, January 2004, pp. 263-270.

[2] D. Gottlieb and S. A. Orszag, *Numerical Analysis of Spectral Methods: Theory and Applications*, Society for Industrial and Applied Mathematics, Philadelphia, 1977.

[3] S. A. Orszag, "Spectral methods for problems in complex geometries," *Journal of Computational Physics*, vol. 37, 1980, pp.70-92.

[4] Q. H. Liu, "The PSTD algorithm: A time-domain method requiring only two cells per wavelength," *Microwave and Optical Technology Letters*, vol. 15, no. 3, 1997, pp. 158-165.

[5] B. Yang, D. Gottlieb, and J. S. Hesthaven, "Spectral simulation of electromagnetic wave scattering," *Journal of Computational Physics*, vol. 134, 1997, pp. 216-230.

[6] B. Yang, J. S. Hesthaven, "A pseudospectral method for time-domain computation of electromagnetic scattering by bodies of revolution," *IEEE Trans. on Antennas and Propagation*, vol. 47, no. 1, January 1999, pp. 132-141.

[7] L. Lustman, "The time evolution of spectral discretizations of hyperbolic systems," *SIAM Journal of Numerical Analysis*, vol. 23, no. 6, 1986, pp.1193-1198.

[8] D. Gottlieb, L. Lustman, E. Tadmor, "Stability analysis of spectral methods for hyperbolic initial-boundary value systems," *SIAM Journal of Numerical Analysis*, vol. 24, no 2, 1987, pp.241-256.

[9] D. Kosloff, H. Tal Ezer, "A modified Chebyshev Pseudospectral method with an O(N-1) time step restriction," *Journal of Computational Physics*, vol. 104, 1993, pp.457-469.

[10] Z. Gang, Z. Y. Qing, Q. H. Liu, "The 3-D multidomain pseudospectral time-domain method for wideband simulation," *IEEE Microwave and Wireless Components Letters*, vol. 13, no. 5, May 2003, pp. 184-186.

[11] L. N. Trefethen, *Spectral Methods in Matlab*, SIAM, Philadelphia, 2000.

[12] B. Fornberg, *A Practical Guide to Pseudospectal Methods*, Cambridge University Press, 1996.