On the Class of A_1 Numbers: Definition, Numerical Modeling, Domain of Existence and Application

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Abstract

The relation $A_1 = A_{1-} - A_{1+}$ is exploited to advance the class of positive real numbers A_1 by means of the ones A_{1-} and A_{1+}. The latter are defined with the help of the positive purely imaginary zeros of a complex Kummer function of specially picked out parameters. A method is elaborated for counting the numerical equivalents of A_{1-} and A_{1+} (of A_1). Tables and graphs illustrate the outcomes of computations. The limits of the existence domains of each of the three kinds of numbers are found, too. A formula for calculation of the differential phase shift produced by the circular waveguide with azimuthally magnetized ferrite which sustains normal TE_{01} mode is suggested, employing the quantities A_1.

1. Introduction

The differential phase shift provided by the azimuthally magnetized circular and coaxial ferrite waveguides for normal TE_{0n} modes can directly be computed from their parameters through the classes of real A, B and C numbers [1-3]. The complicated way in which they have been determined [1] and the insufficient information for them [1,2] however, hampers their usage in practice. Here, a new simpler definition of the subclass A_1 of the more general A one is given, utilizing the zeros of complex Kummer function [4]. A special iterative scheme is developed, allowing to get the values of numbers. The results are presented numerically and graphically. The application of quantities A_1 is discussed.

2. Definitions

Definition 1: *Let* $\zeta_{k,n}^{(c)}$ *is the* n *th positive purely imaginary zero in* x *of the Kummer confluent hypergeometric function* $\Phi(a,c;x)$ *with* $a = c/2 - jk$ *– complex,* $c = 3$, $x = jz$ *– positive purely imaginary,* z *– real, positive,* k *– real,* $-\infty < k < +\infty$ *and* $\nu_{1,n}$ *is the* n *th zero of Bessel function* $J_1(z/2)$, ($n = 1,2,3,...$). *Let in addition* $K_{1\pm}(c,n,k_\pm) = |k_\pm|\zeta_{k_\pm,n}^{(c)}$ *and* $M_{1\pm}(c,n,k_\pm) = |a_\pm|\zeta_{k_\pm,n}^{(c)}$, *and the pair* $\{|\alpha|, \bar{r}_0\}$ *of positive real numbers* $|\alpha| < 1$ *and* $\bar{r}_0 > \nu_{1,n}$ *satisfies the requirement*:

$$\nu_{1,n} < \bar{r}_0 \sqrt{1-\alpha^2} < L_1(c,n)/|\alpha| \tag{1}$$

in which $L_1(c,n)$ *are positive real numbers, determined by the relation*:

$$L_1(c,n) = \lim_{k_- \to -\infty} K_{1-}(c,n,k_-) = \lim_{k_- \to -\infty} M_{1-}(c,n,k_-)\,. \tag{2}$$

Then, $A_{1\pm} = A_{1\pm}(c,|\alpha|,\bar{r}_0,n)$ *numbers are named the real positive quantities, defined in parametric form through the equations*:

$$A_{1\pm}(|\alpha|,k_\pm) = (2|k_\pm|/|\alpha|)\{(1-\alpha^2)/[(2k_\pm)^2 + \alpha^2]\}^{1/2}, \tag{3}$$

$$\bar{r}_0(c,n,|\alpha|,k_\pm) = [K_{1\pm}(c,n,k_\pm)/(2|k_\pm||\alpha|)]\{[(2k_\pm)^2 + \alpha^2]/(1-\alpha^2)\}^{1/2} \tag{4}$$

with $|\alpha|$ *and* $k_\pm$ *as parameters.* (*The subscripts* "+" *and* "–" *are pertinent to* $k_+ > 0$ *and* $k_- < 0$, *resp.*)

Definition 2: $A_1 = A_1(c,|\alpha|,\bar{r}_0,n)$ *numbers are called the real quantities*:

$$A_1 = A_{1-} - A_{1+} \tag{5}$$

where A_{1-} *and* A_{1+} *are fixed for the same* $|\alpha|$, n *and for the values of* k_+, k_-, $K_{1+}(c,n,k_+)$ *and* $K_{1-}(c,n,k_-)$, *meeting the relation* $K_{1-}(c,n,k_-)/K_{1+}(c,n,k_+) = \{[1+(|\alpha|/(2|k_-|))^2]/[1+(|\alpha|/(2|k_+|))^2]\}^{1/2}$ *which answers to the condition* $\bar{r}_{0+} = \bar{r}_{0-} = \bar{r}_0$. ($A_{1-}$, A_{1+}, k_+, k_-, $K_{1+}(c,n,k_+)$, $K_{1-}(c,n,k_-)$, c, $|\alpha|$, $\bar{r}_0$, n *are regarded in the sense of* Definition 1.)

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Table 1. Values of the numbers A_{1-} and A_{1+} as a function of the parameters $\bar{r}_0$ and $|\alpha|$.

$A_{1\pm}$ / $\bar{r}_0$	A_{1-}	A_{1+}	A_{1-}	A_{1+}	A_{1-}	A_{1+}	A_{1-}	A_{1+}	A_{1-}	A_{1+}
	$\|\alpha\|=0.1$		$\|\alpha\|=0.2$		$\|\alpha\|=0.3$		$\|\alpha\|=0.4$		$\|\alpha\|=0.5$	
4.0	2.95960677	2.44599365	1.31832552	0.80431998						
5.0	6.55546314	6.14456060	3.26632065	2.85512538	2.11237303	1.70069973	1.48182029	1.06949908	1.04141734	0.62830127
6.0	7.80391468	7.46148359	3.89224233	3.54953161	2.54214074	2.19897631	1.82707996	1.48330714	1.35782872	1.01332174
7.0	8.45775545	8.16423062	4.21439089	3.92059018	2.75791768	2.46367188	1.99414482	1.69930792	1.50160542	1.20606611
8.0	8.85147557	8.59462875	4.40600910	4.14888431	2.88431916	2.62674803	2.09013681	1.83197785	1.58208183	1.32323391
9.0	9.10881816	8.88049720	4.52993038	4.30132543	2.96505413	2.73599496	2.15056612	1.92091291	1.63187157	1.40153048
10.0	9.28676010	9.08125846	4.61477047	4.40897609	3.01970616	2.81344498	2.19095900	1.98409081	1.66468843	1.45712458
11.0	9.41504898	9.22821648	4.67535144	4.48821542	3.05831010	2.87069121	2.21915414	2.03091025	1.68730801	1.49835405
12.0	9.51059935	9.33932334	4.70046303	4.54845446	3.08648775	2.91439431	2.23949640	2.06675606	1.70343215	1.52996216
13.0	9.58365506	9.42554120	4.75389676	4.59545356	3.10760066	2.94863503	2.25456176	2.09492419	1.71523200	1.55484105

Definition 3: *The quantities in the sense of* Definitions 1 *and* 2 *for certain* $|\alpha|$, *corresponding to* $k_+=0$ *and to specific* $k_-=k_{c-}\neq 0$ *for which* $\bar{r}_{0+}=\bar{r}_{0c-}=\bar{r}_{0cr}$, $\bar{r}_{0cr}(c,n,|\alpha|,k_+)=\nu_{1n}/(1-\alpha^2)^{1/2}$ *are termed critical numbers and are denoted by the subscripts* "*cr*" *and* "*c–*". *It holds*:

$$A_{1cr}=A_{1c-}\,, \tag{6}$$

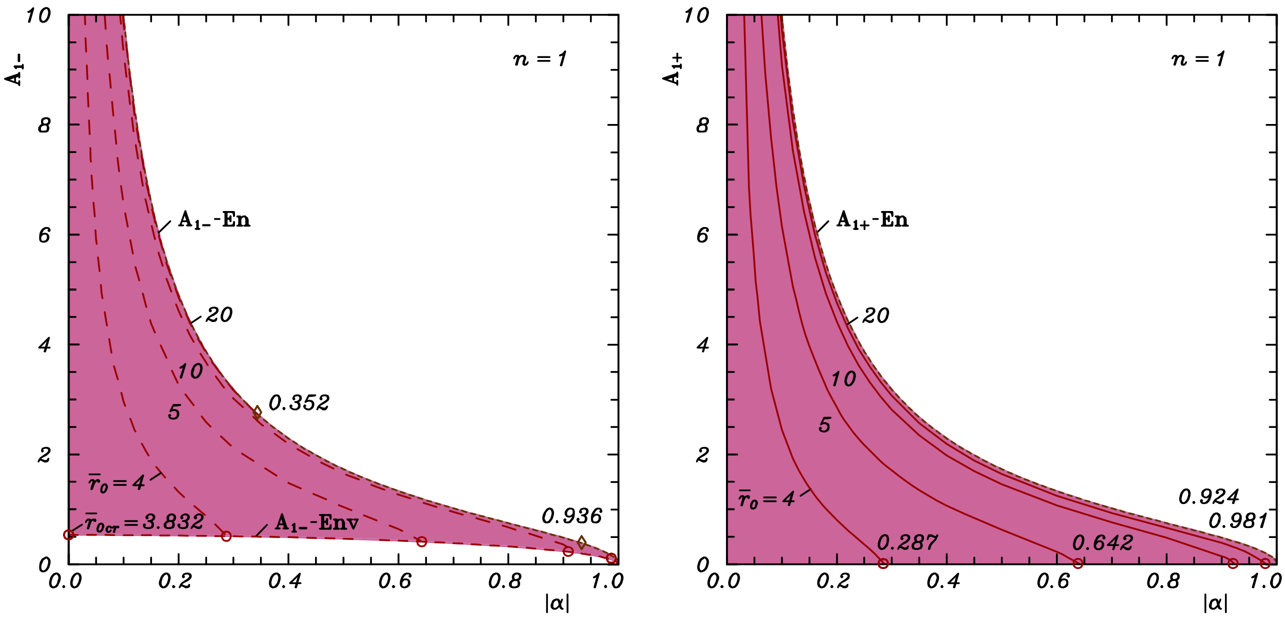
$$A_{1cr+}=0\,. \tag{7}$$

Definition 4: *The quantities in the sense of* Definitions 1 *and* 2 *for given* $|\alpha|$, *relevant to* $k_+\to+\infty$ *and* $k_-\to-\infty$ *for that* $\bar{r}_0=\bar{r}_{0+}$, $\bar{r}_{0+}\to+\infty$, *and* $\bar{r}_0=\bar{r}_{0en-}$, $\bar{r}_{0en-}(c,n,|\alpha_{en-}|,k_-)=L_1(c,n)/[|\alpha_{en-}|(1-\alpha_{en-}^2)^{1/2}]$, $(|\alpha|=|\alpha_{en-}|)$ *are styled limiting numbers and are distinguished by the subscript* "*lim*". *It is fulfilled*:

$$A_{1\lim\pm}=(1-\alpha^2)^{1/2}/|\alpha|\,, \tag{8}$$

$$A_{1\lim}=A_{1\lim-}-A_{1+}\,, \tag{9}$$

($A_{1\lim+}\neq A_{1\lim}$). In particular $L_1(c,n)=6.59365\ 41068$ [2]. (In Refs. [2,5] the symbol $L(c,n)$ stands for $L_1(c,n)$.)



(a) (b)

Figure 1. (a): A_{1-} and (b): A_{1+} numbers vs. $|\alpha|$ in the interval $\langle 0\div 1.0\rangle$ with $\bar{r}_0$ as parameter for $n=1$.

3. Numerical Modeling. Domain of Existence

An $\{|\alpha|,\bar{r}_0\}$ pair, subject to the criterion (1) is chosen. For an arbitrarily picked out value of the parameter k the positive purely imaginary zeros $\zeta_{k,n}^{(c)}$ of the Kummer function for $a=c/2-jk$, $c=3$ and $x=jz$ are found. Then the numerical equivalents of quantities α, k and $\zeta_{k,n}^{(c)}$ are put in the expressions (3) and (4). The counted value of A_{1+} (A_{1-}) is accepted as the one searched for, if the determined in the same way value of the parameter $\bar{r}_0$ coincides with its own initially selected one within the boundaries of the set in advance accuracy. Otherwise, k is changed and the procedure is repeated. The calculations are performed twice for $k_+>0$ and $k_-<0$, yielding the numerical equivalents of numbers A_{1+} and A_{1-}, resp. of A_1. The results of computations are presented in Tables 1 and 2 and in Figs. 1 and 2, assuming $n=1$. As seen, for identical $\bar{r}_0$ and $|\alpha|$, $A_{1-}>A_{1+}$, ($A_1>0$). Besides, at fixed $\bar{r}_0$ the quantities A_1 are almost constant with respect to $|\alpha|$. For $\{|\alpha|,\bar{r}_0\}$ pairs which do not submit the rule (1) $k_\pm$, $\zeta_{k_\pm,n}^{(c)}$ and $A_{1\pm}$ (A_1) do not exist and for this reason the relevant positions in Tables 1 and 2 are blank (the relevant regions in Figs. 1 and 2 are portrayed in white). The functions $A_{1c-}=A_{1c-}(|\alpha|)$ and $A_{1cr}=A_{1cr}(|\alpha|)$ ($A_{1\lim\pm}=A_{1\lim\pm}(|\alpha|)$ and $A_{1\lim}=A_{1\lim}(|\alpha|)$), featured by the $A_{1-}-Env$ – and A_1-Env – dashed curves in Figs. 1a and 2, resp. (the $A_{1-}-En$ –, $A_{1+}-En$ – and A_1-En – dotted lines in Figs. 1a, 1b and 2, resp.), depict the boundaries of the domain of existence of the A_{1-}, A_{1+} and A_1 numbers (the violet areas in Figs. 1a and 1b, and the pink one in Fig. 2).

4. Application

The zeros $\zeta_{k,n}^{(c)}$ of $\Phi(a,c;x)$ with values of the parameters, given in Definition 1 specify the eigenvalue spectrum $\bar{\beta}_2=\zeta_{k,n}^{(c)}/(2\bar{r}_0)$ for normal TE_{0n} modes of the circular waveguide of radius r_0, entirely filled with azimuthally magnetized lossless remanent ferrite, characterized by a scalar permittivity $\varepsilon=\varepsilon_0\varepsilon_r$ and a Polder permeability tensor of off-diagonal element $\alpha=\gamma M_r/\omega$, (γ – gyromagnetic ratio, M_r – remanent magnetization, ω – angular frequency of the wave), provided $x=jz_0$, $z_0=2\bar{\beta}_2\bar{r}_0$, $k=\alpha\bar{\beta}/(2\bar{\beta}_2)$, $\bar{\beta}=\beta/(\beta_0\sqrt{\varepsilon_r})$, $\bar{\beta}_2=\beta_2/(\beta_0\sqrt{\varepsilon_r})$, $\bar{r}_0=\beta_0 r_0\sqrt{\varepsilon_r}$, ($\beta$ – phase constant of the wave, $\beta_2=[\omega^2\varepsilon_0\mu_0\varepsilon_r(1-\alpha^2)-\beta^2]^{1/2}$ – radial wavenumber, $\beta_0=\omega\sqrt{\varepsilon_0\mu_0}$ – free space phase constant) [5]. Since $\bar{\beta}_\pm(\alpha_\pm,k_\pm)=\{(1-\alpha_\pm^2)/[1+(\alpha_\pm/(2k_\pm))^2]\}^{1/2}$ [5], (3) takes the form:

$$A_{1\pm}=\bar{\beta}_\pm/|\alpha|. \tag{10}$$

Table 2. Values of the numbers A_1 as a function of the parameters $\bar{r}_0$ and $|\alpha|$.

$\bar{r}_0$ \ $\lvert\alpha\rvert$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
4.0	0.51361313	0.51393519							
4.5	0.45655129	0.45685602	0.45735534	0.45803591	0.45887795				
5.0	0.41090254	0.41119527	0.41167330	0.41232121	0.41311607	0.41402616			
5.5	0.37355418	0.37383887	0.37430225	0.37492686	0.37568678	0.37654605	0.37745676		
6.0	0.34243109	0.34271073	0.34316443	0.34377282	0.34450697	0.34532680	0.34617882		
6.5	0.31609670	0.31637355	0.31682144	0.31741903	0.31813452	0.31892366	0.31972747	0.32046952	
7.0	0.29352483	0.29380071	0.29424580	0.29483690	0.29553931	0.29630471	0.29706856	0.29774735	
7.5	0.27396300	0.27423934	0.27468406	0.27527212	0.27596596	0.27671317	0.27744369	0.27806689	
8.0	0.25684682	0.25712479	0.25757113	0.25815896	0.25884792	0.25958149	0.26028408	0.26085790	
8.5	0.24174470	0.24202527	0.24247486	0.24306482	0.24375195	0.24447566	0.24515477	0.24568434	
9.0	0.22832097	0.22860495	0.22905917	0.22965321	0.23034109	0.23105812	0.23171749	0.23220704	0.23238741
9.5	0.21631061	0.21659869	0.21705870	0.21765850	0.21834933	0.21906237	0.21970516	0.22015827	0.22027379
10.0	0.20550163	0.20579438	0.20626118	0.20686819	0.20756385	0.20827523	0.20890415	0.20932383	0.20937783
10.5	0.19572240	0.19602032	0.19649478	0.19711023	0.19781238	0.19852412	0.19914150	0.19953033	0.19952562
11.0	0.18683249	0.18713602	0.18761889	0.18824389	0.18895396	0.18966784	0.19027570	0.19063590	0.19057488
11.5	0.17871592	0.17902544	0.17951737	0.18015289	0.18087218	0.18158975	0.18218989	0.18252337	0.18240806
12.0	0.17127600	0.17159184	0.17209343	0.17274034	0.17346999	0.17419266	0.17478664	0.17509508	0.17492721
12.5	0.16443155	0.16475401	0.16526578	0.16592485	0.16666590	0.16739492	0.16798415	0.16826900	0.16805005
13.0	0.15811386	0.15844320	0.15896562	0.15963756	0.16039095	0.16112745	0.16171319	0.16197573	0.16170698

If it is assumed that the quantities throughout the paper have the sense, ascribed to them in this Section, then (1) expresses the condition for phase shifter operation of the configuration [5] and in view of (10) and (5), the differential phase shift produced $\Delta\overline{\beta} = \overline{\beta}_{-} - \overline{\beta}_{+}$ [5] could be figured out from the formula:

$$\Delta\overline{\beta} = A_1 |\alpha| . \qquad (11)$$

For example, when $\bar{r}_0 = 4$ and $|\alpha| = 0.1$, $A_1 = 0.5136\ 1313$ (cf. Table 2), it is obtained $\Delta\overline{\beta} = 0.0513\ 6131$. Moreover, the symbols, labeled by the subscripts "*cr*" and "*c–*", and the $A_{1-}-Env$ – and A_1-Env – curves in Figs. 1a, 2, resp., the circles on them, correspond to the cut-off state of the geometry. The letters, marked by the indices (index) "*lim–*" and "*en–*" ("*lim+*") are related with the envelopes in the phase diagram, appearing in case of negative magnetization [5] and with the $A_{1-}-En$ – and A_1-En – lines in Figs. 1a and 2, resp. with the rhombs on them (are pertinent to infinitely large frequencies in case of positive magnetization [5] and to the $A_{1+}-En$ – curve in Fig. 2b).

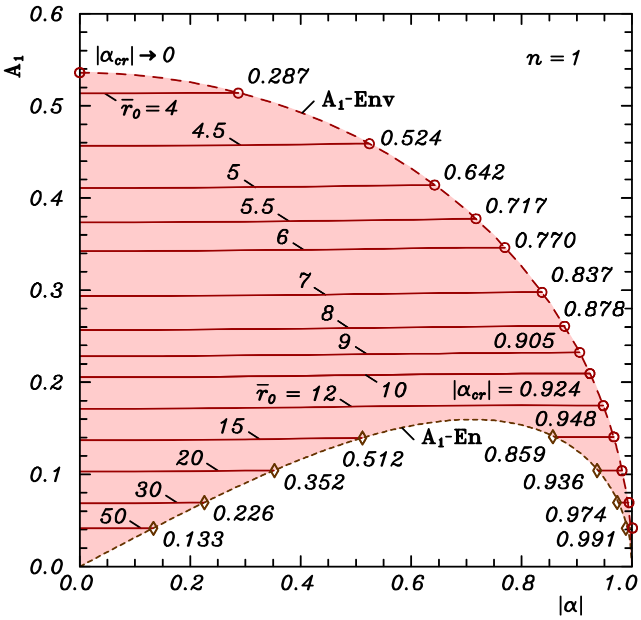


Figure 2. A_1 numbers vs. $|\alpha|$ in the interval $\langle 0 \div 1.0 \rangle$ with $\bar{r}_0$ as parameter for $n = 1$.

5. Conclusion

Definitions of the classes of positive real numbers A_{1-}, A_{1+} and A_1 are given in terms of the positive purely imaginary zeros of a certain complex Kummer function. The values of quantities in question are modelled numerically, utilizing a reiterative technique and are represented in a numerical and graphical form. The boundaries of the domains of their existence are specified. The theory of waveguides is pointed out as a field of putting into practice of the results of analysis.

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7. References

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