Rigorous Approach to Analysis of 2-D Electrostatic- Field Problems for Multi- Conductor Systems

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Abstract

The two-dimensional $N-$body potential problem is rigorously solved with emphasis on electrostatics. By the Method of Analytical Regularization, the solution is obtained from N coupled second kind infinite systems of linear algebraic equations that are effectively solved numerically by a truncation method. A combination of the fast convergence with fast and accurate computation of the matrix elements, based on the Fast Fourier Transform, makes the computational routine extremely efficient: in most cases the computational time measures a few seconds. As an example, the developed algorithm is applied to a calculation of the capacitance matrix for N arbitrary profiled charged cylinders in a space surrounded by a grounded cylindrical shield.

1. Introduction

Two-dimensional boundary value problems for Laplace's equation describe many problems of practical interest arising in aero- and hydro-dynamics, electrostatics, elasticity theory, etc. [1]. For example, studies of physical phenomena in two-dimensional metals and thin films often lead to mixed electrostatic boundary-value problems [2], and similar problems arise when studying $N-$cylinder electrostatic lenses and propagation of the TEM modes in micro-strip transmission lines [3]. When the contour of a conductor coincides with the co-ordinate surface of one of the coordinate systems in which the Laplace's equation is separable, the Fourier method (method of separation of variables) is used. More generally, a variety of potential problems have been solved by the conformal mapping method. These results are described in many classical handbooks and monographs. The number of such solved problems is highly restricted. Nowadays the practical need for simulation of devices used in practice requires the development of more universal methods to tackle problems with objects of various finite-width shapes. One of such numerous examples is the capacitance calculation for thick electrodes [4] where a physically reasonable meaning of "edge capacitance" arises only because an accurate charge distribution over the whole electrode could not be accurately calculated.

In order to address this difficulty we present a new semi-analytical approach to the analysis of 2-D electrostatic-field problems for multi-conductor systems. The problem to be solved is treated as a classical Dirichlet boundary value problem for Laplace's equation.

It is well-known that in 2-D electrostatics the basic formula describing the potential distribution for a single conductor with contour S is given by $U(q)=(1/2\pi)\int_S \log|p-q|Z(p)\,dL_p$, where $Z(p)$ is the line charge density and $G(p,q)=\frac{1}{2\pi}\log|p-q|$ is the Green's function for Laplace's equation in 2-D space. If the contour S is charged to a potential value U_0 then $Z(p)$ may be found by solving the equation $\frac{1}{2\pi}\int_S \log|p-q|\,Z(p)\,dS_p=U_0,\ q\in S$. This equation is a first kind Fredholm equation with a singular kernel; it is ill-posed. Nevertheless, this problem has been tackled by many authors who used the direct numerical schemes for solving its discrete analogue in a form of a first kind algebraic equation. Theoretically, any numerical method applied to solve this equation is unable to guarantee uniform convergence, or pre-determined computational accuracy. The only way to avoid these shortcomings is to transform the initial equation into a second kind Fredholm equation discretization of which guarantees uniform convergence and any pre-determined accuracy of the numerical solution depending on truncation number. This is the objective of this paper. We employ the Method of Analytical Regularization (MoR), in particular, described in [5] - [7]. In this paper we generalize the MoR for potential $N-$body problem where each body is an arbitrary profiled cylinder.

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2. Regularization of the problem: MoR

Consider $N-1$ arbitrary profiled charged perfect electric conductor cylinders embedded into a homogeneous dielectric medium with relative permittivity ε_r. The finite dielectric medium is bounded by the infinitesimally thin grounded cylindrical shell. The problem is to find electrostatic potential U elsewhere inside the shielded region. This electrostatic problem is fully described by the Dirichlet boundary value problem for Laplace's equation

$$\Delta U = 0 \tag{1}$$

with boundary conditions given at the surface of each of N cylinders: $U|_{S_n} = V_n$, $n = 1 \ldots N-1$; $U|_{S_N} = 0$. To employ the regularization procedure all contours S_n must be smooth enough and non-self crossing to provide their continuous parameterization and twice differentiation at each point of S_n. The main challenge of this problem is that all the conductors are arbitrary-shaped and the classical separation of variables method is not applicable here. We use a more general approach based on an integral representation. Using the superposition principle we seek solution for the total field potential U as the sum of the single-layer potentials contributed by each cylinder:

$$U(q) = \sum_{j=1}^{N} \int_{S_j} G(|p-q|) Z_j(p) dl_p; \tag{2}$$

where Z_j is the unknown line charge density of the j-th conductor, S_j is the boundary contour of the j-th conductor, and points q lie in the area between the contours. Kernel G of the integral equation (2) is the two-dimensional free space Green's function $G = \frac{1}{2\pi}\log(|p-q|)$ where $|p-q|$ is the distance between points q and p. Applying boundary conditions to the equation (2) one can arrive at the coupled system of integral equations for the unknowns Z_j:

$$\sum_{j=1}^{N} \int_{S_j} G_{ij}(|p-q|) Z_j(p) dl_p = -u_i(q),\ q \in S_i,\ i = 1 \ldots N \tag{3}$$

Equation (3) represents a system of the first kind Fredholm integral equations that are generally ill-posed. In this paper equation (3) is first transformed into a second kind Fredholm equation by analytical means, and then numerical methods (truncation) are applied. The technique used for this purpose employs the idea of analytical regularization [5-7]. For this purpose the kernel of the initial equation is separated into a suitable singular part and a smooth remainder. Then the singular operator is analytically inverted. Finally integral equations are transformed into an infinite system of the linear algebraic equations of the second kind.

After parameterization of the contours and introducing new notations: $V_s(\theta) = -u_s(\eta(\theta))$; $l(\theta) = \sqrt{x'(\theta)^2 + y'(\theta)^2}$; $R_{sj}(\theta,\tau) = |p-q| = |\eta_s(\theta) - \eta_j(\tau)| = \{[x_s(\theta) - x_j(\tau)]^2 + [y_s(\theta) - y_j(\theta)]^2\}^{1/2}$ we obtain the system of N integral equations:

$$\sum_{j=1}^{N} \int_{-\pi}^{\pi} G(R_{sj}(\theta,\tau)) z_j(\tau) d\tau = V_s(\theta),\ s = 1, 2 \ldots N. \tag{4}$$

For $G(R_{sj}(\theta,\tau))$ such that $s \neq j$ points corresponding to θ and τ belong to different contours and so $R_{sj}(\theta,\tau) \neq 0$ everywhere; hence, the corresponding integral terms do not contain singularities. For $G_{ss}(\theta,\tau)$ the corresponding integral contains singularity of logarithmic type at the points In this case we analytically separate the Green's function into the singular part, representing singularity on a circle, and a regular part L^{sj} that does not contain any singularity:

$$
\begin{aligned}
&2\pi G(R_{sj}(\theta,\tau)) = \log(R_{sj}(\theta,\tau)) = L^{sj}(\theta,\tau), \quad s \neq j, \\
&2\pi G(R_{sj}(\theta,\tau)) = \log(R_{sj}(\theta,\tau)) = L^{sj}(\theta,\tau) + \log(2\sin\left|(\theta-\tau)/2\right|), \quad s = j.
\end{aligned} \tag{5}
$$

Now we can determine L^{sj} from (5). The function $L^{sj}, s = j$ is a regular function, defined everywhere except at points $\theta = \tau$; the function $L^{sj}, s \neq j$ is defined everywhere. It can be shown that for the Laplace's equation this regular function has the same degree of smoothness as the contour parameterization. An exact expression for $L^{sj}, s = j$ at the points of singularity where $\theta = \tau$ was obtained analytically: $L^{s,j}(\theta,\tau) = \frac{1}{2\pi}\log(l(\theta))$, where $l(\theta) = \sqrt{x(\theta)^2 + y(\theta)^2}$ is an arc length. Now we can redefine function $L^{sj}, s = j$ to be analytical everywhere.

Using well-known Fourier expansion, we can formulate expression for the singular part of Green's function:

$$
\log\left(2\sin\left|\frac{\theta-\tau}{2}\right|\right) = \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^{in(\theta-\tau)}}{|n|}, \tag{6}
$$

We expand the regular function L^{sj} it into double Fourier series: $L^{sj}(\theta,\tau) = \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} l_{nm}^{sj} e^{i(n\theta+m\tau)}$. Also the unknown function z_j and the given potential function are represented by their Fourier series: $z_j(\tau) = \sum_{-\infty}^{\infty}\xi_n^j e^{in\tau}$; $V_j(\theta) = \sum_{-\infty}^{\infty} v_n^j e^{in\theta}$. After substitution of all expansions into (3) one can arrive at the system of N integral equations. Using orthogonal properties and completeness of the functions and rescaling unknown Fourier coefficients of charge density function ξ_n^s as follows: $\tilde{\xi}_n^s = \sigma^{-1}\xi_n^s$, $\sigma_n = |n|^{1/2}$ when $n \neq 0$ and $\sigma_0 = 1$, we obtain infinite system of linear algebraic equations of the second kind:

$$
\tilde{\xi}_n^s(1-\delta_{n0}) + \sum_{j=1}^{N}\sum_{m=-\infty}^{\infty}\sigma_n\sigma_m l_{n,-m}^{sj}\tilde{\xi}_m^j = \sigma_n v_n^s, \; n = 0,\pm1,\pm2...; \; s = 1,2,...,N. \tag{7}
$$

Fourier expansions are calculated numerically as all functions are regular. The infinite system can be effectively solved by a truncation method. The solution of the truncated system monotonically and rapidly converges to the exact solution. The above solution automatically incorporates the reciprocal influence of two charged cylinders, allowing accurate calculation of the line charge densities on the both boundaries and then the field potentials at any point of the space between the conductors.

3. Numerical results

As illustration of the effectiveness of the obtained solution we calculate the capacitance matrix for the assembly of arbitrary profiled cylinders located inside the grounded shield. There are no limitations on number of cylinders with arbitrary smooth cross-section. The high efficiency of the code is also the result of enforcement of the discrete Fast Fourier Transform. This makes filling of the matrix very fast routine procedure. For example, the computation time for a problem with the four inner cylinders and truncation number $N_{tr} = 256$ does not exceed 4.5 seconds on a standard PC. Efficiency of the developed method is illustrated also by the behaviour of the normalized truncation error versus truncation number N_{tr} shown in Fig.1 for the structure with a circular shield of radius 1 with embedded elliptic conductor with major semi-axis $b_1 = 0.5$ and various values of the minor semi-axis b_2.

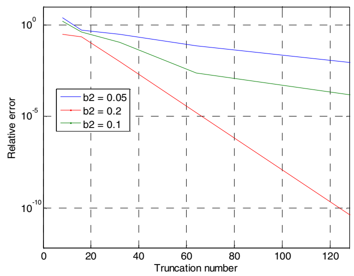


Fig.1. Normalized truncation error versus N_{tr}

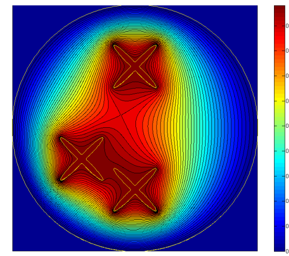
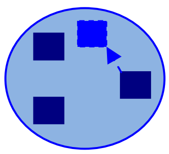


Fig.2. Conceptual shielded transmission line

The power of the method is illustrated in Fig.2, where the distribution of the electrostatic field is shown for some conceptual four conductor shielded transmission line. The profile of each conductor is described by the super-ellipse equation [8]. In conclusion we reproduce the calculations for the capacitance matrix calculated by formula $C_{i,j} = Q_i / u_j$ in the case of the circular shield and conductors of nearly rectangular cross-section.



Symmetric location:

$$C = \begin{pmatrix} 1.3351 & 1.3351 & 1.3351 \\ 1.3349 & 1.3349 & 1.3349 \\ 1.3349 & 1.3349 & 1.3349 \end{pmatrix}$$

Non-symmetric location:

$$C = \begin{pmatrix} 1.0421 & 1.0421 & 1.0421 \\ 0.9751 & 0.9751 & 0.9751 \\ 1.4500 & 1.4500 & 1.4500 \end{pmatrix}$$

4. Conclusion

The two-dimensional $N-$body potential problem for Laplace's equation with Dirichlet boundary conditions is rigorously solved by the Method of Analytical Regularization. The problem is transformed to a numerical analysis of an infinite system of linear algebraic equations of the second kind This explains its fast convergence and guaranteed computational accuracy, depending only upon truncation number N_{tr}. The only limitation imposed on the contour is its smoothness. When the contour incorporates corners they should be rounded. The computational time is of few seconds on a standard PC. This means that developed algorithm may be used when solving the optimization problems, for example, for optimization of multi-conductor transmission lines.

5. References

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