Wave Tomography of Time-Varying Disordered Structures

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Abstract

An original model based on the first principles is constructed for the temporal correlation of wave fields propagating in random scattering media. The predictions of this model are consistent, in general, with those of the diffusing-wave spectroscopy. It is shown that considering the wave vector as a free parameter that can vary at will, one can provide an additional dimension to the data, which results in a tomographic-type reconstruction of the full space-time dynamics of a complex structure, instead of a plain spectroscopic technique. In the Fourier space the problem is reduced to a spherical mean transform defined for a family of spheres containing the origin and therefore is easily invertible.

1. Introduction

Temporal correlation of waves scattered off or transmitted through a random medium has been for a long time a very efficient tool in studying space-time dynamics of a great variety of complex structures [1-3]. At the microscale, typical examples include the solutions of macromolecules and colloidal suspensions, gels, foams, granular materials, biological tissues and laboratory plasma. Macroscopic structures range from atmospheric turbulence, heterogeneous Earth and internal waves in the ocean to ionospheric and interstellar plasmas. The basic quantity measured usually in such experiments is the temporal autocorrelation function (ACF) defined as

$$\Gamma(\tau) = \langle u(t)\, u^*(t+\tau) \rangle \ , \tag{1}$$

where $u(t)$ is the complex amplitude of the field, and the angular brackets mean ensemble average. In some situations the analysis is based on the power spectrum of the field fluctuations, $\Gamma(\nu)$, which is given by Fourier transformation of the temporal ACF.

The random medium, in its turn, is typically characterized by some constitutive parameter $\tilde{\varepsilon}(\mathbf{r},t)$. The physical content of $\tilde{\varepsilon}(\mathbf{r},t)$ depends on a specific problem under study: it may be permittivity in electromagnetic applications, slowness for acoustic waves, etc., but will be generally referred to as scattering potential in the sequel. The quantity we are interested in to recover is usually the two-point correlation function $B_\varepsilon(\mathbf{r},\tau)$ of the scattering potential, or a corresponding $(3+1)$ spectral density $\Psi_\varepsilon(\mathbf{K},\nu)$ obtained by Fourier transforming $B_\varepsilon(\mathbf{r},\tau)$ in both space and time.

Two extreme regimes of the wave scattering are usually considered. When the scattering is rather weak, the wave field can be described using a single-scattering (Born) approximation. The resulting theory relates the power spectrum $\Gamma(\nu)$ measured for a wave propagating initially in the direction of vector $\mathbf{k}_i$ and then scattered in the direction of vector $\mathbf{k}_s$, to the value of spectral density $\Psi_\varepsilon(\mathbf{k}_s-\mathbf{k}_i,\nu)$ taken at the Bragg vector $\mathbf{K}=\mathbf{k}_s-\mathbf{k}_i$ [1]. In this regime the selectivity of the wave scattering in the **K**-space allows one, in principle, to reconstruct the full space-time dynamics of the structure by varying both the radiation frequency and the propagation directions of the incident and scattered waves [4].

The condition of weak scattering imposes severe limitations on the disordered system and the radiation used. On the other hand, probing the structures with multiply scattered waves may be much more effective: in principle, they are sensitive to very small displacements in the medium. However, in order to realize this potential, we should have a rather accurate model relating the measured quantity, $\Gamma(\tau)$, to the statistics of the scattering potential. The multiple scattering is usually accounted for by changing totally the paradigm, namely replacing the wave equation as a basic tool with a radiation transfer equation or its asymptotics, a diffusion model for example. One such technique

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called diffusing wave spectroscopy (DWS) was proposed about two decades ago [5,6]. At present, the applications of DWS technique and its derivatives are not limited to studying the dynamics of relatively simple structures like colloidal suspensions, but may be efficiently implemented, for example, in a functional imaging of the brain activity.

Despite a great popularity and plausible results, the critical analysis of DWS shows that it is based on hardly convincing arguments. Actually, DWS is a phenomenological theory, which models the wave transport as a random walk between scatterers. As is known, this model is unsuitable for describing wave propagation along short paths which, in its turn, results in underestimating the field correlation for long times. Nevertheless, the main problem of DWS is in the construction of the ACF which is performed in a purely heuristic way, assigning to each specific path a phase accumulated by the photon on its way from one *point scatterer* to another. As a result, DWS provides us with a somewhat limiting amount of information: in contrast to the single scattering experiments, inversion of the data allows one to estimate only the mean square displacement of the scattering particle instead of the full space-time dynamics of the system which is described by the correlation function $B_\varepsilon(\mathbf{r},\tau)$.

Earlier attempts to analyze the DWS basic principles starting from the wave equation were undertaken in [7,8]. The motivation there was to arrive at the diffusion model based on a diagrammatic technique and to find the necessary corrections for a correlated disorder. In the present paper we attack the problem from another point, constructing an original model of temporal correlations which is based on the first principles and therefore is free of the intrinsic limitations of the DWS phenomenology. Our model is in a sense complementary to that of DWS since it is designed for the description of a moderately strong scattering regime, where both highly scattered diffusing waves, on the one hand, and those propagating mainly in the forward direction, on the other, may coexist.

Two main questions are explored here. First of all, the measured ACF is presented as a subtle integral transform of the spectral density $\Psi_\varepsilon(\mathbf{K},\nu)$, taking into account, albeit approximately, the multiple scattering effects. This allows one, in particular, to understand what is the physical content of the phenomenological theory like DWS. Second and more important, we show that measuring the normalized ACF of the scattered field as a function of both time delay τ and the wave vector $\mathbf{k}$, it is possible to reconstruct the full space-time dynamics of the scattering structure. Unlike all known wave tomography methods (all of them being based on the Born or Rytov approximation), the relevant $\mathbf{K}$-space spectral components of the scattering potential contributing to the measured field constitute a manifold of the same dimensionality as the ambient space. It will be shown, however, that the classical methods of integral geometry are still applicable here. This technique can be applied to noninvasive study of particle motion or density fluctuations in a variety of disordered systems.

2. Temporal correlation

Our starting point is the reduced Helmholtz equation,

$$\nabla^2 u(\mathbf{r},t)+k^2\left[1+\tilde{\varepsilon}(\mathbf{r},t)\right]u(\mathbf{r},t)=-\delta(\mathbf{r}) \,, \tag{2}$$

written for a point source located in an unbounded statistically homogeneous medium. Solving the direct problem means here the evaluation of the temporal ACF $\Gamma(\mathbf{k},\tau)$. Denoting the unknown function as $\Gamma(\mathbf{k},\tau)$ we emphasize the obvious fact that this quantity should depend not only on the time shift τ but also on the wave vector $\mathbf{k}$ directed along the line connecting the source with the observation point.

Apart from a classical diagrammatic technique there exist an alternative perturbative approach based on the path integral formulation for the analysis of Eq. (2). Recently it was used, in particular, to treat the two-frequency mutual coherence function (frequency field-field correlator) in random media [9,10]. As is known, Fourier transformation of this correlator leads to an impulse response function (photon time-of-flight distribution) $J_{\mathbf{k}}(t)$ measured for a short narrowband pulse. The results obtained in Refs. [9,10] reproduce the two-scale structure of the coherence function, which corresponds to ballistic and diffuse components of the scattered field coexisting in the impulse response $J_{\mathbf{k}}(t)$ even in the regime of multiple scattering. Also, rather good agreement with known experimental results (including microwaves, optics, and ultrasound) was demonstrated [10].

It would be natural to extend this path integral approach to the evaluation of the coherence function accounting also for the temporal dynamics of the medium. The generalization is straightforward and may be performed following the procedure described in our previous paper [10]. Omitting the details of the derivation, let us concentrate on the final expression for the normalized ACF, $\tilde{\Gamma}(\mathbf{k},\tau)=\Gamma(\mathbf{k},\tau)/\Gamma(\mathbf{k},0)$. This function demonstrates an exponential decay,

$$\tilde{\Gamma}(\mathbf{k},\tau) = \exp\left[-\tilde{\chi}(\mathbf{k},\tau)\right] \ , \tag{3}$$

where the decrement is given by the spectral expansion

$$\tilde{\chi}(\mathbf{k},\tau) = 2\int_{-\infty}^{\infty} d\nu \left[1-\exp(i\nu\tau)\right]\tilde{\chi}(\mathbf{k},\nu) \ . \tag{4}$$

In its turn, spectrum $\tilde{\chi}(\mathbf{k},\nu)$ entering Eq. (4) has the form

$$\tilde{\chi}(\mathbf{k},\nu) = \frac{\pi}{8}k^3 L\int d\mathbf{K}\,K^{-2}\ \vartheta\left(K-\left|\,2\mathbf{k}\cdot\mathbf{K}/K\,\right|\right)\Psi_\varepsilon(\mathbf{K},\nu) \ , \tag{5}$$

where $\vartheta(x)$ is the Heaviside step function. As for the single-scattering approximation, Eq. (5) is easily mapped onto the Ewald construction, but is not local here: in order to calculate the value of $\tilde{\chi}(\mathbf{k},\nu)$ for fixed values of **k** and ν, we should integrate the spectral density $\Psi_\varepsilon(\mathbf{K},\nu)$ outside the eight curve formed by the two Ewald spheres, with a weighting factor K^{-2}.

Analysis of the temporal ACF given by Eqs. (3)-(5) allows one to arrive at the two main conclusions. First, for a fixed **k** the information content of $\tilde{\chi}(\mathbf{k},\tau)$ is similar to the corresponding ACF decrement in the classical DWS theory. In fact, $\tilde{\chi}(\mathbf{k},\tau)$ may be interpreted as (a half of) the structure function of the phase fluctuations in the measured field; for a definition of structure function see Refs. [1,3]. Moreover, there is another fact that unifies our approach with the DWS technique: both provide us with only a **K**-weighted average, in striking contrast to the weak scattering experiments, where the registered quantity is **K**-selective. Although much more sensitive to small displacements, the multiple scattering and the associated **K**-averaging effect make it impossible to directly invert the measured decrement in order to reconstruct the dynamics of the structure at a specific values of **K**.

The second important observation to be mentioned is that the temporal ACF $\tilde{\Gamma}(\mathbf{k},\tau)$ depends strongly on the wave number, or more precisely, on the wave vector **k** when the random medium is statistically anisotropic. Hence, considering the wave vector **k** as a free parameter that can vary at will, we provide an additional dimension to the data, which may result in a tomographic-type reconstruction of the space-time dynamics instead of a plain spectroscopic technique. We will explore this idea in the next section.

3. Inversion

In order to make the inversion we have to measure the ACF, or, simply its decrement $\tilde{\chi}(\mathbf{k},\tau)$, for waves of different frequencies and a variety of angular orientation of the structure with respect to the straight line connecting the source with the observation point. Indeed, to recover the spectral density $\Psi_\varepsilon(\mathbf{K},\nu)$, we should perform the two-step procedure as follows.

First, we differentiate both sides of Eq. (4) with respect to τ and invert the resulting Fourier transform yielding

$$\tilde{\chi}(\mathbf{k},\nu) = \frac{1}{2\pi\nu}\int_0^{\infty} d\tau \sin(\nu\tau)\,\tilde{\chi}'(\mathbf{k},\tau) \ . \tag{6}$$

Note that in principle this step may be unnecessary if our goal is to recover $\Psi_\varepsilon(\mathbf{K},\nu)$, the temporal correlation function of a given **K**-component of the disorder.

At the second step, having at hand $\tilde{\chi}(\mathbf{k},\tau)$ which can be considered now as a function of **k** for each fixed value of ν, we have to invert the integral transform (5). Performing in Eq. (5) the integration by parts with respect to the absolute value of vector **K**, we arrive at

$$\tilde{\chi}(\mathbf{k},\nu) = \frac{\pi}{4}k^3 L\int d\mathbf{K}\ \delta\left(K-\left|\,2\mathbf{k}\cdot\mathbf{K}/K\,\right|\right)F_\varepsilon(\mathbf{K},\nu) \ , \tag{7}$$

where we have introduced an auxiliary function

$$F_\varepsilon(\mathbf{K},\nu) = K^{1-m}\int_K^{\infty} dK\,K^{m-3}\,\Psi_\varepsilon(\mathbf{K},\nu) \tag{8}$$

(here, for the sake of generality, m is the dimensionality of the problem). In order to find the spectral density $\Psi_\varepsilon(\mathbf{K},\nu)$, provided the auxiliary function is known, we should multiply Eq. (8) by K^{m-1} and then differentiate:

$$\Psi_{\varepsilon}(\mathbf{K},\nu) = -K^{3-m} \frac{d}{dK}\left[K^{m-1} F_{\varepsilon}(\mathbf{K},\nu) \right] . \tag{9}$$

The spectral density $\Psi_{\varepsilon}(\mathbf{K},\nu)$ and then $F_{\varepsilon}(\mathbf{K},\nu)$ are both even functions, so that we can keep the integration over only one Ewald sphere and double the result. Therefore, Eq. (7) is nothing else than a spherical mean of the function $F_{\varepsilon}(\mathbf{K},\nu)$ for a family of spheres containing the origin. These are just the Ewald spheres constructed for different frequencies and propagation directions (the information contained actually in the wave vector **k**). The spherical mean operator is known to be invertible [11,12]. In particular, it may be converted into the classical Radon transform using a geometric inversion of the **K**-space with respect to a reference sphere centered at the origin [11].

4. Summary

In this paper we have developed a new concept of analyzing the space-time dynamics of time-varying random structures. The solution of the direct problem obtained has a clear physical interpretation and provides us with a rather strong insight into the nature of the DWS-like theories. In contrast to the DWS technique, however, our results have been derived from the first principles, without using any phenomenological parameters such as diffusion constant. Our theory is valid for situations where the size of the scattering system is comparable to the transport mean-free-path, i.e., it is suitable to fill the gap between ballistic regime well described by the single-scattering approximation, on the one hand, and the diffusion regime, on the other. The better accuracy is predicted for long and intermediate time scales (relatively short paths) where the diffusion approximation fails to describe the wave transport correctly.

Concerning the inversion, the developed concept extends the conventional diffraction tomography technique to the multiple scattering regime. Indeed, we have arrived at the tomographic-type reconstruction based after all on a classical Radon transform. In principle, this technique allows for recovering full space-time dynamics of the structure under study. The disordered systems are not required to be composed of identical particles, statistically anisotropic and fractal structures can be probed equally well, with no *a priori* information needed.

References

1. V. I. Tatarskii, *Effects of the Turbulent Atmosphere on Wave Propagation*, Moscow, Nauka, 1967.
2. B. J. Berne and R. Pecora, *Dynamic Light Scattering*, New York, Wiley, 1976.
3. A. Ishimaru, *Wave Propagation and Scattering in Random Media*, New York, Academic, 1978.
4. E. Wolf, "Principles and development of diffraction tomography," in A. Consortini (ed.), *Trends in Optics*, San Diego, Academic, 1996, pp. 83-110.
5. G. Maret and P. E. Wolf, "Multiple light scattering from disordered media. The effect of Brownian motion of scatterers," *Z. Phys. B* **65**, 1987, pp. 409-413.
6. D. J. Pine, D. A. Weitz, P. M. Chaikin, and E. Herbolzheimer, "Diffusing wave spectroscopy," *Phys. Rev. Lett.* **60**, 1988, pp. 1134-1137.
7. M. J. Stephen, "Temporal fluctuations in wave propagation in random media," *Phys. Rev. B* **37**, 1988, pp. 1-5.
8. F. C. MacKintosh and S. John, "Diffusing-wave spectroscopy and multiple scattering of light in correlated random media," *Phys. Rev. B* **40**, 1989, pp. 2383-2406.
9. G. Samelsohn and V. Freilikher, "Spectral coherence of wave fields in random media," in B. van Tiggelen and S. Skipetrov (eds.), *Wave Scattering in Complex Media: From Theory to Applications*, Dordrecht, Kluwer, 2003, pp. 189-201.
10. G. Samelsohn, V. Freilikher, and M. Haridim, "Spectral coherence and time-domain transport of waves in random media," *Phys. Rev. E* **78**, 2008, 066602.
11. G. Samelsohn, "Diffuse time tomography of strongly scattering random structures," *Waves Random Complex Media* **19**, 2009, pp. 11-27.
12. G. Samelsohn, "Diffuse time tomography of isotropic random media," *Physica B* **405**, 2010, pp. 3026-3028.