Wavelet Analysis for Electromagnetic Field

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Abstract

We obtain an exact integral representation of the free electromagnetic field as a superposition of localized elementary solutions. The representation involves the continuous wavelet transform. The results can be used for studying electromagnetic fields with multiscale structure.

1 Introduction

Phase-space representations of acoustic and electromagnetic fields were recently discussed in several papers [1-5]. These representations are used to expand wave fields in terms of Gaussian beams in the case of frequency domain or of focus wave modes in the case of time domain [6]. The works were mostly based on phase-space approximate representations on windowed Fourier transform with Gabor window. These approximations were used for beam summation representation of half-plane or of edge diffraction [7]. The field radiated by ultrawideband sources was also expanded using this formalism [8]. Our subject is also the phase-space representation of fields which satisfy Maxwell equations. We present the exact representation based on continuous wavelet analysis. The main distinction with the works mentioned above is that instead of using windowed Fourier transform with Gabor window, we use the continuous wavelet transform. Our results allow to expand fields as a superposition of ultrawideband short pulses localized in space instead of focus wave modes. The restrictions on choice of elementary pulses are not very strict and they are presented below. As a possible example of such pulses we suggest to use previously found exact particle-like solution of wave equation [9-10]. This solution is highly localized near the point moving along the straight line with the speed of light. It is exponentially localized both in space and spatial frequency domains. The first representation based on continuous wavelet analysis was obtained by G. Kaiser in [11]. However his representation allows to use only spherically symmetric solutions of very special form with power-law decrease.

2 Integral representation for electromagnetic potential

The aim of this work is to present the decomposition of an electromagnetic field which satisfies the Maxwell equations in the absence of currents and charges as the superposition of localized electromagnetic waves. We consider electromagnetic potentials φ and $\boldsymbol{A} = (A_x, A_y, A_z)$ with the Lorentz gauge condition in the absence of currents and charges. They must satisfy the following equation:

$$\frac{\partial^2}{\partial t^2}\varphi - c^2\Delta\varphi = 0, \tag{1}$$

$$\frac{\partial^2}{\partial t^2}\boldsymbol{A} - c^2\Delta\boldsymbol{A} = 0. \tag{2}$$

This allows to apply the wavelet-based method for the decomposition of scalar wave fields, which was developed in [12-14]. We consider one component of the potential, for instance, A_x. The other components are considered in the same way.

The component A_x satisfies the scalar wave equation. We consider its Fourier transform $\widehat{A}_x(\boldsymbol{k}, t)$ calculated with respect to the spatial coordinates. Due to the general properties of the wave equation it has the form

$$\begin{aligned}\widehat{A}_x(\boldsymbol{k}, t) = \widehat{A}_{x+}(\boldsymbol{k}, 0)\exp(-i|\boldsymbol{k}|ct) \\ +\widehat{A}_{x-}(\boldsymbol{k}, 0)\exp(i|\boldsymbol{k}|ct).\end{aligned} \tag{3}$$

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Then A_x in the coordinate domain can also be represented as the sum of the two terms

$$A_x = A_{x+} + A_{x-}, \tag{4}$$

which has the sense of positive- and negative-frequency parts of A_x. We will consider functions which are square-integrable for a fixed time t.

We now fix an arbitrary square-integrable solution $\psi(\boldsymbol{r}, t)$ of the scalar wave equation, which has only positive-frequency term $\psi = \psi_+$, which is axially symmetric in respect to the x axis and satisfies the following admissibility condition [15]:

$$C_\psi \equiv \int\limits_{\mathbb{R}^3} d^3\boldsymbol{k}\ \frac{|\widehat{\psi}(\boldsymbol{k}, 0)|^2}{|\boldsymbol{k}|^3}\ <\ \infty. \tag{5}$$

We construct the family of elementary pulses $\psi^\nu(\boldsymbol{r}, t)$ by the following formula:

$$\psi^\nu(\boldsymbol{r}, t) = \frac{1}{a^{3/2}} \psi\left(M_{\beta\theta} \frac{\boldsymbol{r} - \boldsymbol{b}}{a},\ \frac{t}{a} \right), \tag{6}$$

where $\nu = (\boldsymbol{b}, a, \beta, \theta)$, $M_{\beta\theta}$ is a rotation matrix by angles $\beta \in [0, 2\pi),\ \theta \in [0, \pi)$, $\boldsymbol{b} \in \mathbb{R}^3$, $a \in (0, \infty)$. An example of the exact solution which satisfies all these conditions is the Gaussian wave packet, found in [9-10] and studied in [16]:

$$\psi_{packet}(\boldsymbol{r}, t) = \frac{1}{x - ct + i\varepsilon} \exp\left[-p\sqrt{1 - \frac{i\theta}{\gamma}} \right], \tag{7}$$

$$\theta = x - ct + \frac{y^2 + z^2}{x - ct + i\varepsilon}, \tag{8}$$

where p, γ, ε are free positive parameters. This solution is an exponentially localized wave packet, moving along x axis with speed c. However the choice of an elementary pulse is not restricted to Gaussian packet and we can use an arbitrary solutions which satisfies the conditions above.

We define the coefficients of the decomposition as follows:

$$W[A_{x\pm}](\nu) = \int\limits_{\mathbb{R}^3} d^3\boldsymbol{r}\ A_{x\pm}(\boldsymbol{r}, t)\, \overline{\psi^\nu(\boldsymbol{r}, \pm t)}, \tag{9}$$

where $W[\ ]$ denotes the wavelet transform [15] of the function in the square brackets. This expression does not depend on time t.

The integral representation of A_x in terms of localized solutions ψ^ν then reads

$$\begin{aligned} A_1(\boldsymbol{r}, t) = \frac{1}{C_\psi} \int d\mu(\nu)\ \{ & W[A_{1+}](\nu) \psi^\nu(\boldsymbol{r}, t) \\ & + W[A_{1-}](\nu) \psi^\nu(\boldsymbol{r}, -t) \}, \end{aligned} \tag{10}$$

where

$$d\mu(\nu) \equiv d^3\boldsymbol{b}\, \frac{da}{a^4}\, d\beta\, d\theta\, \sin\theta. \tag{11}$$

The same results can be obtained for other components of the electromagnetic potential $\boldsymbol{A}$.

3 Integral representation of the electromagnetic field

We base on the results from the previous section to obtain the decomposition of the wave field by using the decomposition of the potential. We introduce the following notation:

$$\boldsymbol{W}[\boldsymbol{A}](\nu) \equiv (W[A_x](\nu), W[A_y](\nu), W[A_z](\nu)),$$

$$\Psi^\nu(\boldsymbol{r},t) = \nabla\psi^\nu(\boldsymbol{r},t). \tag{12}$$

By using this notation we write the integral representation of the electromagnetic field:

$$\boldsymbol{E} = -\int d\mu(\nu)\ \Big\{\Psi^\nu(\boldsymbol{r},t)\,W[\varphi](\nu) + \frac{\partial}{\partial t}\,\psi^\nu(\boldsymbol{r},t)\,\boldsymbol{W}[\boldsymbol{A}](\nu)\Big\}, \tag{13}$$

$$\boldsymbol{B} = \int d\mu(\nu)\,\Psi^\nu(\boldsymbol{r},t)\times\boldsymbol{W}[\boldsymbol{A}](\nu). \tag{14}$$

We write down such expression for positive- and negative-frequency components. Their sum provide the full field.

It is also possible to modify the method described above. We can use different elementary solutions ψ to decompose each component of the electromagnetic potential. Then the elementary electromagnetic pulse Ψ will have the anisotropy, which can be useful for some class of problems.

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