Wave reflection from an inhomogeneous layer with random irregularities

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Abstract

The problem of wave reflection and scattering from an inhomogeneous layer with random irregularities has been solved using the double weighted Fourier transform and the Fock proper-time method. The solution obtained in a small-angle approximation agrees with the geometrical optics approximation, the Rytov approximation and the phase screen method. However, this approach can be exploited without small-angle approximation conditions. And finally, the results of the proposed approach take the form of strict results for the layer with the linear permittivity profile and enable us to account for random inhomogeneities in such a layer.

1. Introduction

Reflectometry – receiving information on an object from reflected or back-scattered signals – is one of the main methods for studying inhomogeneous plasma of ionosphere and nuclear fusion. At present, the theories of wave reflection from the plane-stratified medium and scattering by irregularities in a free space are well-developed. However, waves are often concurrently reflected from an inhomogeneous layer and scattered by irregularities in this layer both in the ionosphere and in the nuclear fusion plasma. These irregularities should be taken into consideration to improve the accuracy of plasma layer diagnostics and study the role of random irregularities in plasma processes.

It is best to take account of the random irregularities in wave reflection using the geometrical optics approximation, when the size of irregularities exceeds the Fresnel radius. But this approach can not be applied to strong field fluctuations associated with random focusing and multipath propagation. The Rytov method incorporates diffraction effects, but only for weak fluctuations. The phase screen method considers both diffraction effects and strong fluctuations, but only if the irregularity is located near some plane.

The double weighted Fourier transform method (DWFT) [1-2] takes into account not only all the above-listed effects, but also allows the spatial processing which improves the signal resolution. However, DWFT deals with the parabolic equation, i.e. with the small-angle approximation, that is inapplicable to the analysis of reflected or back-scattered signals. The analysis is especially complicated when waves are scattered by irregularities located in a region, where the refractive index tends to zero.

In [3-4], the combination of the Fock proper-time method [5] and DWFT was employed to derive the wave field expression without small-angle approximation conditions. Here we use this approach to examine the wave reflection from the inhomogeneous layer with random irregularities.

2. The combination of DWFT and Fock proper-time method

Let us consider the field of a harmonic point source situated at a point $\mathbf{r}_0 = \{z_0, x_0, y_0\} = \{z_0, \boldsymbol{\rho}_0\}$ in the inhomogeneous medium with dielectric permittivity $\varepsilon(\mathbf{r}) = 1 + \varepsilon_1(\mathbf{r})$. This wave (Green function) is described by the Helmholtz equation

$$\Delta G(\mathbf{r}, \mathbf{r}_0) + k^2 \varepsilon(\mathbf{r}) G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0), \tag{1}$$

where $k = \omega / c = 2\pi / \lambda$; c and λ are the velocity of light and the wavelength in a free space, ω is the emission frequency.

The Fock proper-time method [5] involves reducing (1) to the parabolic equation through transformation

$$G(\mathbf{r}, \mathbf{r}_0) = -i / (2k) \int_0^{\tau} U(\mathbf{r}, \mathbf{r}_0, \tau) d\tau . \tag{2}$$

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It is easy to show that $U(\mathbf{r},\mathbf{r}_0,\tau)$ satisfies the following equation

$$2ik\frac{\partial U}{\partial \tau}+\Delta U+k^2\varepsilon(\mathbf{r})U=0 \tag{3}$$

with the initial condition

$$U\big|_{\tau=0}=\delta(\mathbf{r},\mathbf{r}_0). \tag{4}$$

By solving (3)–(4) with DWFT [1-2] and substituting the solution into (2), we get:

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0)=&\frac{4}{k}e^{\frac{i3\pi}{4}}\left(\frac{2\pi}{k}\right)^{3/2}\int_0^\infty d\tau\tau^{3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3s d^3p A(\mathbf{s},\mathbf{p},\tau)\\&\times\exp\left\{ik\left[\tau/2+(\mathbf{r}+\mathbf{r}_0)^2/(2\tau)+2(\mathbf{sp}\tau-\mathbf{sr_0}-\mathbf{pr})+\Phi_1(\mathbf{s},\mathbf{p},\tau)\right]\right\}.\end{aligned} \tag{5}$$

Here

$$\Phi_1(\mathbf{s},\mathbf{p},\tau)=1/2\int_0^\tau\varepsilon_1\left[\mathbf{p}(\tau-\tau')+\mathbf{s}\tau'+\mathbf{r}_+(\tau')+\mathbf{r}_-(\tau')\right]d\tau'+2\int_0^\tau\mathbf{p}_+(\tau')\mathbf{p}_-(\tau')d\tau', \tag{6}$$

$$A(\mathbf{s},\mathbf{p},\tau)=\left(\frac{k}{2\pi}\right)^6\exp\left\{-\frac{1}{8}\int_0^\tau\frac{\partial^2}{\partial\mathbf{r}_+\partial\mathbf{p}}\varepsilon_1\left[\mathbf{p}(\tau-\tau')+\mathbf{s}\tau'+\mathbf{r}_+(\tau')-\frac{\partial}{\partial\mathbf{p}}\Phi_1(\mathbf{s},\mathbf{p},\tau')\right]d\tau'\right\}, \tag{7}$$

$\mathbf{r}_+(\tau')$ and $\mathbf{r}_-(\tau')$ are the solutions to identical ray equations

$$\frac{d\mathbf{r}_{+,-}(\tau')}{d\tau'}=\mathbf{p}_{+,-}(\tau'), \tag{8}$$

$$\frac{d\mathbf{p}_{+,-}(\tau')}{d\tau'}=\frac{1}{4}\frac{\partial}{\partial\mathbf{r}_+}\varepsilon_1\left[\mathbf{p}(\tau-\tau')+\mathbf{s}\tau'+\mathbf{r}_+(\tau')+\mathbf{r}_-(\tau')\right], \tag{9}$$

but with initial conditions specified at different points:

$$\mathbf{r}_+(\tau')\big|_{\tau'=\tau}=\mathbf{p}_+(\tau')\big|_{\tau'=\tau}=0,\quad \mathbf{r}_-(\tau')\big|_{\tau'=0}=\mathbf{p}_-(\tau')\big|_{\tau'=0}=0. \tag{10}$$

To take into account the effect of random inhomogeneities $\tilde{\varepsilon}_1(\mathbf{r})$ against the background of the regular inhomogeneity $\bar{\varepsilon}_1(\mathbf{r})$, we should insert $\varepsilon_1(\mathbf{r})=\bar{\varepsilon}_1(\mathbf{r})+\tilde{\varepsilon}_1(\mathbf{r})$ into (6)-(7), (9) and consider that $\bar{\varepsilon}_1(\mathbf{r})>>\left|\tilde{\varepsilon}_1(\mathbf{r})\right|$.

3. A layer with the linear permittivity profile

As an example of application of the above algorithm let us consider the case when the background medium is a flat layer with the permittivity profile

$$\bar{\varepsilon}(z)=1+\bar{\varepsilon}_1(z)=1-\varepsilon' z. \tag{11}$$

Then from (5)–(7) and the perturbation theory we obtain:

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0)=&\frac{4}{k}e^{\frac{i3\pi}{4}}\left(\frac{k}{2\pi}\right)^{9/2}\int_0^\infty d\tau\tau^{-9/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d^3\xi d^3\xi_0\\&\times\exp\left\{ik\left[\tau/2+(\mathbf{r}-\mathbf{r}_0)^2/(2\tau)+2(\xi_0-\mathbf{r_0})(\xi-\mathbf{r})/\tau-\varepsilon'\tau(z_0+z)/4-\varepsilon'^2\tau^3/96+\tilde{\Phi}_1(\xi,\xi_0,\tau)\right]\right\},\end{aligned} \tag{12}$$

where

$$\tilde{\Phi}_1(\xi,\xi_0,\tau)=1/2\int_0^\tau\tilde{\varepsilon}\left[\xi_0(1-\tau'/\tau)+\xi\tau'-\mathbf{e}_z\varepsilon'(\tau'-\tau)\tau'/4\right]d\tau'. \tag{13}$$

Given $\varepsilon' = 0$, (12) yields results of [3-4]:

$$G(\mathbf{r},\mathbf{r}_0) = \frac{4}{k} e^{\frac{i3\pi}{4}} \left(\frac{k}{2\pi}\right)^{9/2} \int_0^\infty \frac{d\tau}{\tau^{9/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^3\xi d^3\xi_0 \exp\left\{ik\left[\frac{\tau}{2} + \frac{(\mathbf{r}-\mathbf{r}_0)^2}{2\tau} + 2\frac{(\xi_0-\mathbf{r}_0)(\xi-\mathbf{r})}{\tau} + \tilde{\Phi}_1(\xi,\xi_0,\tau)\right]\right\}, \quad (14)$$

where

$$\tilde{\Phi}_1(\xi,\xi_0,\tau) = 1/2\int_0^\tau \tilde{\varepsilon}\left[\xi_0(1-\tau'/\tau) + \xi\tau'\right] d\tau' . \quad (15)$$

As is shown in [4], at $k\left|\tilde{\Phi}_1\right| << 1$ (14) takes the form of the Born approximation, hence our solution (12) incorporates diffraction effects and back scattering. In the small-angle approximation, (14) gives DWFT results [1-2]

$$G(\mathbf{r},\mathbf{r}_0) = G(z,\boldsymbol{\rho};z_0,\boldsymbol{\rho}_0) = \frac{-Ak^2}{4(\pi(z-z_0))^3} \exp\left\{ik(z-z_0) + ik(\boldsymbol{\rho}-\boldsymbol{\rho}_0)^2/(2(z-z_0))\right\}$$
$$\times \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\varsigma d^2\varsigma_0 \exp\left\{2ik(\boldsymbol{\rho}-\varsigma)(\boldsymbol{\rho}_0-\varsigma_0)/(z-z_0) + ik\tilde{\Phi}_1(\varsigma,\varsigma_0)\right\}, \quad (16)$$

where

$$\tilde{\Phi}_1(\varsigma,\varsigma_0) = 0.5\int_{z_0}^{z} \tilde{\varepsilon}\left\{\left[\varsigma(z'-z_0) + \varsigma_0(z-z')\right]/(z-z_0), z'\right\} dz' . \quad (17)$$

Formula (16) fits well the results of the Rytov approximation for weak fluctuations and of the phase screen method for strong fluctuations [1]. Thus, our equation (12) accounts for diffraction effects and strong fluctuations. Moreover, with random fluctuations of the trajectory in integral (17) the scintillation index derived from (16) fits the results of the path integral method [4].

Without random irregularities, i.e. at $\tilde{\varepsilon}_1(\mathbf{r}) = 0$, we can take the integrals over ξ_0, ξ in (12) and obtain

$$G(\mathbf{r},\mathbf{r}_0) = \frac{e^{\frac{i3\pi}{4}}}{4\pi}\left(\frac{k}{2\pi}\right)^{1/2} \int_0^\infty d\tau\tau^{-3/2} \exp\left\{ik\left[\tau/2 + (\mathbf{r}-\mathbf{r}_0)^2/(2\tau) - \varepsilon'\tau(z_0+z)/4 - \varepsilon'^2\tau^3/96\right]\right\}. \quad (18)$$

Formula (18) coincides with the strict solution [6] obtained by other methods. Unlike [6], our approach enables investigation into effects of random inhomogeneities on a reflected wave.

4. Conclusion

This paper offers an algorithm for solving the problem of wave propagation, reflection and scattering in a randomly inhomogeneous medium. In the case of inhomogeneous medium with the linear permittivity profile, the result of the asymptotic method coincides with the strict result. We may, therefore, assume that if the geometrical optics approximation starts from the problem of wave propagation in homogeneous medium, then the proposed approach starts from the problem of wave propagation in the linear profile layer. Derived for the reflected signal is the formula which enables investigations into various statistical moments for signals reflected from a randomly inhomogeneous layer.

Besides, the results obtained may be exploited to develop a spatial signal processing algorithm providing measurements with the super Fresnel resolution under conditions of multipath propagation and strong fluctuations. This processing has the form of analogous DWFT [2, 7-8], but with due regard to reflection and back scattering. This procedure may be carried out using the multiple input - multiple output system employed in communication systems (e.g. [9]). Contrary to the usual system there is a need in a synchronized system in this case that, of course, complicates the signal processing.

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6. References

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