Computation of the dyadic Green's function for electrically and magnetically anisotropic media

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Abstract

A homogeneous non-dispersive electrically and magnetically anisotropic media, characterized by a symmetric positive definite permittivity and permeability tensors are considered. An analytic method for deriving the time-dependent dyadic Green's function (DGF) in these anisotropic media is suggested. This method consists of the following: equations for each column of the DGF are reduced to symmetric hyperbolic system; using the Fourier transform with respect to space variables and matrix transformations we obtain formulae for Fourier images of the DGF columns; finally, the DGF is computed by the inverse Fourier transform. Computational examples confirm the robustness of the suggested method.

1 Introduction

Many technically important materials (media) which become popular in new technologies are anisotropic. For example, the widely used substrate material sapphire and the lithium niobate (LiNbO3), which is used in the design of integrated optics devices, are anisotropic. Anisotropic materials are called homogeneous when their physical properties depend on orientation and do not depend on position. Materials react to applied electromagnetic fields in a variety of ways. For example, if a point pulse source is located in an optical homogeneous isotropic crystal, then fronts of electric and magnetic waves have spherical shapes. The shapes of the fronts in anisotropic materials are not spherical and have very peculiar forms. The simulation of invisible electromagnetic wave phenomena is a very important issue of modern inter-discipline engineering areas.

Analytic methods of DGFs constructions have been studied for isotropic and anisotropic materials in (see, for example, [1-4]). An analytical method for solving IVP for the time-dependent electromagnetic fields in homogenous anisotropic media is studied in [5], [6]. Most of the electromagnetic wave problems have been solved by numerical methods, in particular finite element method, boundary elements method, finite difference method, nodal method (see, for example, [7]).

The time-dependent electric and magnetic fields in homogeneous non-dispersive materials are governed by the following Maxwell's system [8]

$$curl_x\mathbf{H} = \varepsilon_0\overline{\overline{\varepsilon}}\frac{\partial\mathbf{E}}{\partial t} + \mathbf{J},\ \ curl_x\mathbf{E} = -\mu_0\overline{\overline{\mu}}\frac{\partial\mathbf{H}}{\partial t},\ \ div_x(\varepsilon_0\overline{\overline{\varepsilon}}\mathbf{E}) = \rho_e,\ \ div_x(\mu_0\overline{\overline{\mu}}\mathbf{H}) = 0, \tag{1}$$

where $x = (x_1, x_2, x_3) \in R^3$ is a space variable, $t \in R$ is a time variable, $\mathbf{E} = (E_1, E_2, E_3)$ and $\mathbf{H} = (H_1, H_2, H_3)$ are the electric and magnetic fields with components: $E_i = E_i(x,t)$, $H_i = H_i(x,t)$, $i = 1,2,3$; $\mathbf{J} = (J_1, J_2, J_3)$ is the density of the electric current with components $J_i = J_i(x,t)$; ε_0 and μ_0 are the permittivity and permeability of the free space respectively. $\overline{\overline{\varepsilon}} = (\varepsilon_{ij})_{3\times3}$ and $\overline{\overline{\mu}} = (\mu_{ij})_{3\times3}$ are the permittivity and permeability matrices, respectively; ρ_e is the density of electric charges. The electric charges and current are sources of electromagnetic waves. We assume these sources are given. It follows from (1) that electric charges and current have to satisfy the conservation law of charges

$$\frac{\partial\rho_e}{\partial t} + div_x\mathbf{J} = 0. \tag{2}$$

In this paper we suppose that

$$\mathbf{E} = 0,\ \ \mathbf{H} = 0,\ \ \mathbf{J} = 0,\ \ \rho_e = 0,\ \ \texttt{for}\ \ t < 0. \tag{3}$$

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Remark 1. The first two equations of (1) under conditions (3) imply the last two equalities of (1) (we assume here that (2) is satisfied).

2 Maxwell's equations as a first order symmetric hyperbolic system

The first and second equations of (1) can be written as the following first order symmetric hyperbolic system

$$\frac{1}{c}A_0\frac{\partial \mathbf{V}}{\partial t}+\sum_{j=1}^{3}A_j\frac{\partial \mathbf{V}}{\partial x_j}=\mathbf{F} \tag{4}$$

where $c=1/\sqrt{\varepsilon_0\mu_0}$ is the light velocity, $\mathbf{V}=(\mathbf{E},\mathbf{H})$, $\mathbf{F}=(-\mathbf{J},0_{3\times 1})$, $A_0=\begin{pmatrix}\sqrt{\frac{\varepsilon_0}{\mu_0}}\overline{\overline{\varepsilon}} & 0\\ 0 & \sqrt{\frac{\mu_0}{\varepsilon_0}}\overline{\overline{\mu}}\end{pmatrix}_{6\times 6}$,

$$A_j=\begin{pmatrix}0_{3\times 3} & A_j^1\\ (A_j^1)^* & 0_{3\times 3}\end{pmatrix}_{3\times 3},\ A_1^1=\begin{pmatrix}0&0&0\\0&0&1\\0&-1&0\end{pmatrix},\ \ A_2^1=\begin{pmatrix}0&0&-1\\0&0&0\\1&0&0\end{pmatrix},\ \ A_3^1=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&0\end{pmatrix}.$$

3 Equations for the time-dependent DGF of electrically and magnetically anisotropic media

The DGF for the time-dependent Maxwell's equations (1) is defined as a matrix $\mathbf{G}(x,t)$ of the order 6×3 whose columns $(\mathbf{E}^m,\mathbf{H}^m)=(E_1^m,E_2^m,E_3^m,H_1^m,H_2^m,H_3^m)$, $m=1,2,3$ satisfy

$$curl_x\mathbf{H}^m \;=\; \varepsilon_0\overline{\overline{\varepsilon}}\frac{\partial \mathbf{E}^m}{\partial t}+\mathbf{e}^m\delta(x)\delta(t),\ \ curl_x\mathbf{E}^m=-\mu_0\overline{\overline{\mu}}\frac{\partial \mathbf{H}^m}{\partial t},\ \ \mathbf{E}^m\mid_{t<0}=0,\ \ \mathbf{H}^m\mid_{t<0}=0, \tag{5}$$

Here $\mathbf{e}^1=(1,0,0)$, $\mathbf{e}^2=(0,1,0)$, $\mathbf{e}^3=(0,0,1)$ are basis vectors of the Cartesian coordinates; $\delta(x)=\delta(x_1)\delta(x_2)\delta(x_3)$ is the Dirac delta function of the space variable concentrated at $x_1=0,\ x_2=0,\ x_3=0$; $\delta(t)$ is the Dirac delta function of the time variable concentrated at $t=0$.
Elements of the m-th column $(\mathbf{E}^m,\mathbf{H}^m)$ of the DGF are the electric and magnetic fields in the considered anisotropic medium arising from the pulse dipole $\mathbf{e}^m\delta(x)\delta(t)$.

Similar to Section 2 equalities (5) can be written as

$$\frac{1}{c}A_0\frac{\partial \mathbf{V}^m}{\partial t}+\sum_{j=1}^{3}A_j\frac{\partial \mathbf{V}^m}{\partial x_j}=\mathcal{E}^m\delta(x)\delta(t),\ \ \mathbf{V}^m\mid_{t<0}=0,\ \ m=1,2,3, \tag{6}$$

where $\mathcal{E}^m=(-\mathbf{e}^m,0,0,0)$ is such vector with six components that the first three components are components of $-\mathbf{e}^m$ and other components are equal to zero.

4 Deriving formulae for electric and magnetic fields

Let $\tilde{\mathbf{V}}^m(\nu,t)=(\tilde{V_1}^m(\nu,t),\tilde{V_2}^m(\nu,t),\tilde{V_3}^m(\nu,t),\tilde{V_4}^m(\nu,t),\tilde{V_5}^m(\nu,t),\tilde{V_6}^m(\nu,t))$ be the Fourier image of $\mathbf{V}^m(x,t)$ with respect to $x=(x_1,x_2,x_3)\in R^3$, i.e. for $j=1,2,3$

$$\tilde{V_j}^m(\nu,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}V_j^m(x,t)e^{ix\cdot\nu}dx_1dx_2dx_3,\ \ \nu=(\nu_1,\nu_2,\nu_3)\in R^3,\ \ x\cdot\nu=x_1\nu_1+x_2\nu_2+x_3\nu_3,\ \ i^2=-1.$$

Equalities (6) can be written in terms of $\tilde{\mathbf{V}}^m(\nu,t)$ as follows

$$A_0\frac{\partial\tilde{\mathbf{V}}^m}{\partial t}-icB(\nu)\tilde{\mathbf{V}}^m=c\mathcal{E}^m\delta(t),\ \ \tilde{\mathbf{V}}^m(\nu,t)|_{t<0}=0, \tag{7}$$

where $B(\nu)=(\nu_1A_1+\nu_2A_2+\nu_3A_3)$.

Diagonalization A_0 and $B(\nu)$ simultaneously. The matrix A_0 is symmetric positive definite and $B(\nu)$ is symmetric. In this step we construct a non-singular matrix $\mathcal{T}(\nu)$ and a diagonal matrix $D(\nu)=diag(d_k(\nu),\ \ k=1,2,...,6)$ with real valued elements such that

$$\mathcal{T}^*(\nu)A_0\mathcal{T}(\nu)=I,\ \ \mathcal{T}^*(\nu)B(\nu)\mathcal{T}(\nu)\ \ =\ \ D(\nu), \tag{8}$$

where $\mathbf{I}$ is the identity matrix, $\mathcal{T}^*(\nu)$ is the transposed matrix to $\mathcal{T}(\nu)$.

Computing $\mathcal{T}(\nu)$ and $D(\nu)$ can be made by the following way: we find $A_0^{-1/2}$ and then using the matrix $A_0^{-1/2}B(\nu)A_0^{-1/2}$ we construct $\mathcal{T}(\nu)$ and $D(\nu)$.

Finding $A_0^{-1/2}$. For the given positive definite matrix A_0 we compute an orthogonal matrix $\mathcal{R}$ by the eigenfunctions of A_0 such that $\mathcal{R}^*A_0\mathcal{R}=\mathcal{L}$, where $\mathcal{R}^*$ is the transpose matrix to $\mathcal{R}$ and $\mathcal{L}=diag(\lambda_k,\ \ k=1,2,...,9)$ is the diagonal matrix with positive elements λ_k which are eigenvalues of A_0. The matrix $\mathcal{L}^{1/2}$ is defined by the formula $\mathcal{L}^{1/2}=diag(\sqrt{\lambda_k},\ \ k=1,2,...,9)$ and $A_0^{1/2}$ is defined by $A_0^{1/2}=\mathcal{R}\mathcal{L}^{1/2}\mathcal{R}^*$. The matrix $A_0^{-1/2}$ is the inverse to $A_0^{1/2}$.

Finding $\mathcal{T}(\nu)$ and $\mathbf{D}(\nu)$. Let matrix $B(\nu)$ be given and matrix $A_0^{-1/2}$ be found. Let us consider the matrix $A_0^{-1/2}B(\nu)A_0^{-1/2}$ which is symmetric with real valued elements. The diagonal matrix $D(\nu)$ is constructed by eigenvalues of $A_0^{-1/2}B(\nu)A_0^{-1/2}$. The columns of the orthogonal matrix $Q(\nu)$ are formed by normalized orthogonal eigenfunctions of $A_0^{-1/2}B(\nu)A_0^{-1/2}$ corresponding to eigenvalues $d_k(\nu),\ \ k=1,2,...,9$. The matrix $\mathcal{T}(\nu)$ is defined by the formula $\mathcal{T}(\nu)=A_0^{-1/2}Q(\nu)$.

Deriving a solution of IVP (7). Let $\mathbf{D}(\nu)$ and $\mathcal{T}(\nu)$, satisfying (8), be constructed. We find a solution of (7) in the form $\tilde{\mathbf{V}}^m(\nu,t)=\mathcal{T}(\nu)\mathbf{Y}^m(\nu,t)$, where $\mathbf{Y}^m(\nu,t)$ is unknown vector function. Substituting $\tilde{\mathbf{V}}^m(\nu,t)=\mathcal{T}(\nu)\mathbf{Y}^m(\nu,t)$ into (7) and then multiplying the obtained vector differential equation by $\mathcal{T}^*(\nu)$ and using (8) we find

$$\frac{\partial\mathbf{Y}^m}{\partial t}-icD(\nu)\mathbf{Y}^m=c\mathcal{T}^*(\nu)\mathcal{E}^m\delta(t),\ \ t\in R,\ \ \mathbf{Y}^m(\nu,t)|_{t\le 0}=0. \tag{9}$$

Using the ordinary differential equations technique a solution of this initial value problem (9) is given by

$$\mathbf{Y}^m(\nu,t)=c\theta(t)\left[\cos(cD(\nu)t)+i\sin(cD(\nu)t)\right]\mathcal{T}^*(\nu)\mathcal{E}^m,$$

where $\theta(t)$ is the Heaviside function, i.e. $\theta(t)=1$ for $t\ge 0$ and $\theta(t)=0$ for $t<0$; $\cos(cD(\nu)t)$ and $\sin(cD(\nu)t)$ are diagonal matrices whose diagonal elements are $\cos(cd_k(\nu)t)$ and $\sin(cd_k(\nu)t),\ \ k=1,2,...,6$, respectively.

A solution of (7) is determined by $\tilde{\mathbf{V}}^m(\nu,t)=c\theta(t)\mathcal{T}(\nu)\left[\cos(cD(\nu)t)+i\sin(cD(\nu)t)\right]\mathcal{T}^*(\nu)\mathcal{E}^m$. Finally the vector function $\mathbf{V}(\nu,t)$ satisfying (6) can be found by the inverse Fourier transform

$$\mathbf{V}^m(x,t)=\frac{1}{(2\pi)^3}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}\tilde{\mathbf{V}}^m(\nu,t)e^{-i\nu\cdot x}d\nu_1d\nu_2d\nu_3. \tag{10}$$

5 Conclusion

In the paper a novel efficient method of constructing the time-dependent DGF for electrically and magnetically anisotropic homogeneous media has been described. This method is based on matrix computations and the inverse Fourier transform which is done numerically. This method combines in a rational way analytical techniques and numerical computations. Using this method we have generated images of electric and magnetic fields components which are a result of the electromagnetic radiations arising from a pulse dipole with a fixed polarization in different electrically and magnetically anisotropic homogeneous media. Examples of images of electric and magnetic fields in the bi-axial anisotropic material are presented in Fig.1 and Fig.2. The 3D surfaces plots of $z = E_2^3(x_1, x_2, -\sqrt{1/3}x_1, 1/c)$ and $z = H_2^3(x_1, x_2, -\sqrt{1/3}x_1, 1/c)$ are given in Fig.1 and Fig.2. The magnetic permeability and dielectric permittivity for this crystal are given as follows: $\mu_0 diag(1, 1, 1)$ (henry/meter) ($\mu_0 = 1.257 \times 10^{-6}$), $\varepsilon_0 diag(2.25, 1, 0.25)$ (farad/meter) ($\varepsilon_0 = 8.854 \times 10^{-12}$).

6 References

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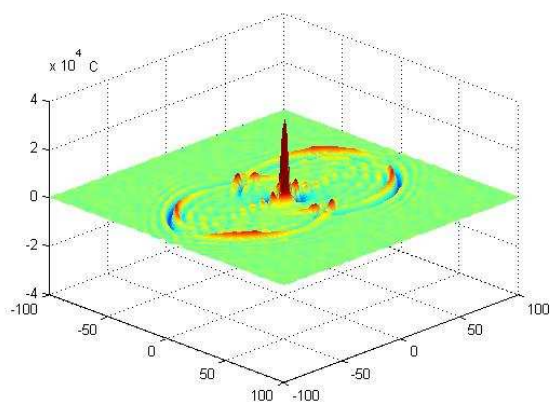

Fig 1.3-D level plot of $E_2^3(x_1, x_2, -\sqrt{1/3}x_1, 1/c)$.

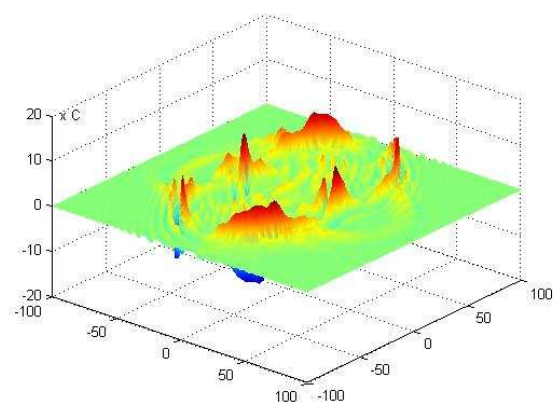

Fig 2.3-D level plot of $H_2^3(x_1, x_2, -\sqrt{1/3}x_1, 1/c)$.