Circuit Models in Quantum Electrodynamics

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Abstract

Network methods are applied to establish quantum mechanical models of distributed circuits. Based on the Hamiltonian description of the Foster equivalent circuits a quantization of the equivalent circuits is performed. The quantum mechanical interaction of the modes via nonlinear elements is discussed. As an example the DC–pumped Josephson parametric amplifier is treated quantum mechanically.

1 Introduction

Nanoscale electric and electronic circuits operated at extremely low energy levels, especially when dealing with single-electron transport and single–photon excitation, are becoming subject to quantum mechanical considerations. The quantum theory of circuits already has been addressed by B. Yurke [1]. It has been shown that linear reciprocal lossless electromagnetic structures can be represented by canonical Foster equivalent circuits. Analytic methods, e.g. Green's function or mode matching approaches, or numerical methods in connection with system identification techniques provide for the synthesis of lumped element models [2, 3]. In the following we present a method to construct quantum mechanical models of electromagnetic structures.

2 The Foster Representation of an Electromagnetic Structure coupled to a Lossless Nonlinear Circuit

As has been shown in [2, 3], for linear reciprocal lossless (LRLL) electromagnetic structures canonical Foster multiport equivalent circuits can be established. These equivalent circuit representations are derived from the poles and residues of the rational transfer functions and can either be obtained from analytic solutions of the field problem or, using system identification methods, from numerical solutions.

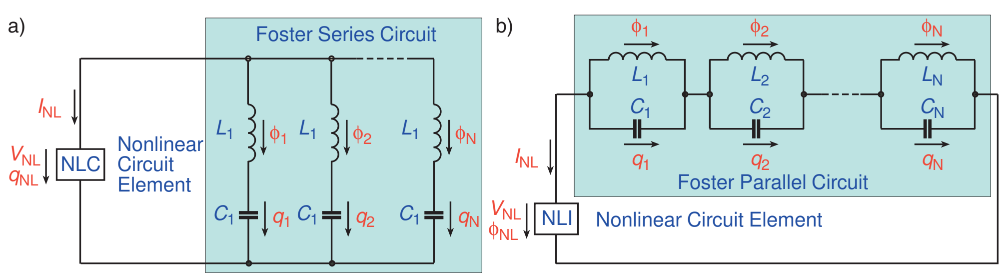

Figure 1: a) Foster Series LC Equivalent Circuit Connected to a Nonlinear Element, b) Foster Parallel LC Equivalent Circuit Connected to a Nonlinear Element.

Consider the Foster representation of an LRLL electromagnetic structure shown in Fig. 1 (a). If no nonlinear circuit element NLC is connected to this structure, we have to replace NLC by a short-circuit. In that case the N series resonant circuits formed by the inductors L_n and the capacitors C_n, with $n = 1 \dots N$, are not coupled with each other. Consider the charges $q_n(t)$ of the n^{th} capacitor as the generalized coordinates, the "potential" energy stored in C_n is given by $q_n^2/2C_n$ and the "kinetic" energy stored in L_n is given by $\dot{q}_n^2 L_n/2$. The Lagrange function is defined as the sum of the kinetic energies minus the sum of the

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potential energies and consequently given by

$$\mathcal{L}_{lin} = \sum_{n=1}^{N} \left(\frac{L_n}{2} \dot{q}_n^2 - \frac{1}{2C_n} q_n^2 \right) . \tag{1}$$

Now, let us consider the nonlinear element NLC. We assume this element to be a nonlinear capacitor, where the voltage v_{NL} is a nonlinear function of the charge q_{NL},

$$v_{NL} = f_{CN}(q_{NL}) , \qquad i_{NL}(t) = \frac{dq_{NL}(t)}{dt} . \tag{2}$$

The energy $W_{CN}(t)$ flowing into NLC from time 0 to t is given by

$$W_{CN} = \int_0^t v_{NL}(t) i_{NL}(t) dt = \int_0^t v_{NL}(t) \frac{dq_{NL}(t)}{dt} dt = \int_{q_{NL}(0)}^{q_{NL}(t)} f_{CN}(q_{NL}) dq_{NL} . \tag{3}$$

Under the assumption that the characteristics $f_{CN}(q_{NL})$ is hysteresis-free, the energy W_{CN} is a unique function of q_{NL} and can be written as $W_{CN}(q_{NL})$. With this we obtain the total Lagrangian

$$\mathcal{L} = \mathcal{L}_{lin} + W_{CN} = \frac{L_1}{2} \dot{q}_1^2 + \frac{L_2}{2} \dot{q}_2^2 \cdots + \frac{L_N}{2} \dot{q}_N^2 - \frac{1}{2C_1} q_1^2 - \frac{1}{2C_2} q_2^2 \cdots - \frac{1}{2C_N} q_N^2 + W_{CN}(q_{NL}) . \tag{4}$$

The momenta ϕ_n conjugate to the generalized coordinates are given by

$$\phi_n = \frac{\partial \mathcal{L}}{\partial \dot{q}_n} = L_n \dot{q}_n . \tag{5}$$

Since $\dot{q}_n$ is the current flowing through the inductor L_n the generalized momentum ϕ_n is the magnetic flux stored in L_n. We obtain the Hamilton function $\mathcal{H}$ from

$$\mathcal{H} = \sum_{n=1}^{N} \phi_n f_n - \mathcal{L} . \tag{6}$$

Inserting (4) and (5), this yields the system Hamilton function

$$\mathcal{H}_{NLC} = \sum_{n=1}^{N} \left(\frac{1}{2L_n} \phi_n^2 + \frac{1}{2C_n} q_n^2 \right) + W_{CN}(q_{NL}) , \qquad \text{with} \;\; q_{NL} = q_{NL0} - \sum_{n=1}^{N} q_n , \tag{7}$$

where q_{NL0} is the initial charge in the nonlinear element. In the case of the Foster parallel LC equivalent circuit, depicted in Fig. 1 (b), with an embedded lossless hysteresis-free nonlinear inductor NLI, we can proceed in the dual way and obtain the Hamilton function

$$\mathcal{H}_{NLI} = \sum_{n=1}^{N} \left(\frac{1}{2L_n} \phi_n^2 + \frac{1}{2C_n} q_n^2 \right) + W_{LN}(\phi_{NL}) , \qquad \text{with} \;\; \phi_{NL} = \phi_{NL0} + \sum_{n=1}^{N} \phi_n , \tag{8}$$

where $W_{LN}(\phi_{NL})$ is the energy stored in the nonlinear inductor and ϕ_{NL0} is the initial flux.

3 The Quantization of the Foster Circuit

Knowing the classical Hamilton functions (7) and (8), we obtain the quantum mechanical Hamilton operators by replacing the charge and flux variables q_n and ϕ_n by the respective quantum mechanical operators $\boldsymbol{q}_n$ and $\boldsymbol{\phi}_n$

$$\boldsymbol{H}_{NLC} = \boldsymbol{H}_{lin} + \boldsymbol{W}_{CN}(\boldsymbol{q}_{NL}) , \qquad \text{with} \;\; \boldsymbol{q}_{NL} = \boldsymbol{q}_{NL0} - \sum_{n=1}^{N} \boldsymbol{q}_n , \tag{9a}$$

$$\boldsymbol{H}_{NLI} = \boldsymbol{H}_{lin} + \boldsymbol{W}_{LN}(\boldsymbol{\phi}_{NL}) , \qquad \text{with} \;\; \boldsymbol{\phi}_{NL} = \boldsymbol{\phi}_{NL0} + \sum_{n=1}^{N} \boldsymbol{\phi}_n , \tag{9b}$$

where

$$\boldsymbol{H}_{lin} = \sum_{n=1}^{N} \left(\frac{1}{2L_n} \boldsymbol{\phi}_n^2 + \frac{1}{2C_n} \boldsymbol{q}_n^2 \right) \tag{10}$$

is the Hamilton operator describing the unperturbed linear Foster circuit. The flux operators $\boldsymbol{\phi}_m$ as the generalized position operators and the charge operators as the generalized momentum operators $\boldsymbol{q}_n$ must fulfill the commutator relation $[\boldsymbol{\phi}_m, \boldsymbol{q}_n] = \delta_{mn}$. Introducing the destruction operators $\boldsymbol{a}_n$ and the creation operators $\boldsymbol{a}_n^\dagger$ by

$$\boldsymbol{a}_n = \sqrt{\frac{1}{2\hbar\omega_n L_n}} \boldsymbol{\phi}_n + j \sqrt{\frac{\omega_n L_n}{2\hbar}} \boldsymbol{q}_n \,, \quad \boldsymbol{a}_n^\dagger = \sqrt{\frac{1}{2\hbar\omega_n L_n}} \boldsymbol{\phi}_n - j \sqrt{\frac{\omega_n L_n}{2\hbar}} \boldsymbol{q}_n \,, \tag{11}$$

where † denotes the Hermitian conjugate and $\omega_n = \sqrt{L_n C_n}$, we can write the Hamiltonian in the form

$$\boldsymbol{H}_{lin} = \frac{1}{2} \sum_{n=1}^{N} \hbar\omega_n \left(\boldsymbol{a}_n^\dagger \boldsymbol{a}_n + \boldsymbol{a}_n \boldsymbol{a}_n^\dagger \right) \,. \tag{12}$$

4 The Quantization of a DC–Driven Josephson Circuit

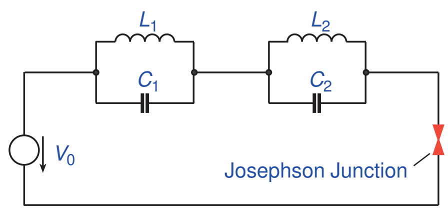

Figure 2: Josephson Junction Connected to two Resonant Circuits.

Figure 2 shows a circuit consisting of two lossless resonant circuits with inductors L_1, L_2 and capacitors C_1, C_2, respectively, and a DC source V_0. With the electric charge q_i in capacitor C_i and the magnetic flux ϕ_i in inductor L_i we can write the classical Hamilton function of the system as [4]

$$\mathcal{H} = \frac{1}{2C_1} q_1^2 + \frac{1}{2L_1} \phi_1^2 + \frac{1}{2C_2} q_2^2 + \frac{1}{2L_2} \phi_2^2 + \frac{1}{2\pi} \phi_0 I_J \left[1 - \cos\left(\omega_0 t + \frac{2\pi(\phi_1 + \phi_2)}{\phi_0} \right) \right] \,, \tag{13}$$

where $\phi_0 = (h/2e_0) \simeq 2.06783461 \cdot 10^{-15}$ Vs is the flux quantum and $\omega_0/2\pi = 2e_0V_0/h \simeq 483.6\ V_0$ GHz/mV is the frequency with which the Josephson junction oscillates due to the applied DC voltage V_0. The quantum mechanical Hamilton operator can be obtained by replacing the classical system variables q_1, q_2, ϕ_1, and ϕ_2 by the quantum mechanical operators $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{\phi}_1$, and $\boldsymbol{\phi}_2$. Introducing the operators defined in (11) and the Hamiltonian $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_1$, with an unperturbed part $\boldsymbol{H}_0$ and a perturbation $\boldsymbol{H}_1$, yields

$$\boldsymbol{H}_0 = \frac{1}{2} \hbar\omega \left(\boldsymbol{a}_1^\dagger \boldsymbol{a}_1 + \boldsymbol{a}_1 \boldsymbol{a}_1^\dagger + \boldsymbol{a}_2^\dagger \boldsymbol{a}_2 + \boldsymbol{a}_2 \boldsymbol{a}_2^\dagger \right) \,, \tag{14a}$$

$$\boldsymbol{H}_1 = W_J \left[1 - \cos\left[\omega_0 t + \kappa_1(\boldsymbol{a}_1 + \boldsymbol{a}_1^\dagger) + \kappa_2(\boldsymbol{a}_2 + \boldsymbol{a}_2^\dagger) \right] \right] \,, \tag{14b}$$

with

$$W_J = \frac{\phi_0 I_J}{2\pi} \,, \quad \kappa_i = \sqrt{\frac{2\alpha Z_i}{\pi Z_{F0}}} \,, \quad \alpha = \frac{e_0^2}{4\pi\hbar\epsilon_0 c_0} \,, \quad Z_i = \sqrt{\frac{L_i}{C_i}} \,, \tag{15}$$

where $\alpha \approx 137.036^{-1}$ is the atomic fine structure constant, $Z_{F0} \approx 377\ \Omega$ is the free space wave impedance, and the impedance Z_i characterizes the i^{th} resonant circuit. $\boldsymbol{H}_0$ is the unperturbed Hamiltonian which describes the photon states in the resonant circuits L_1C_1 and L_2C_2. The interaction between the DC driven Josephson junction and the photonic states is described by the perturbation operator $\boldsymbol{H}_1$. Since the DC voltage V_0 can be stabilized with an arbitrarily large capacitor such that thermal and quantum fluctuations of the DC voltage can be kept arbitrarily low, the DC driven oscillation with frequency ω_0 are treated classically.

4.1 Example: The Josephson Parametric Amplifier

For small amplitudes of $\boldsymbol{a}_1$ we can expand the cosine term in (14b) according to

$$\cos(\omega_0 t + \boldsymbol{a} + \boldsymbol{a}^\dagger) = \cos(\omega_0 t) - \sin(\omega_0 t)(\boldsymbol{a} + \boldsymbol{a}^\dagger) + \frac{1}{2}\cos(\omega_0 t)(\boldsymbol{a} + \boldsymbol{a}^\dagger)^2 + \mathcal{O}^{(3)}. \quad (16)$$

The perturbation Hamiltonian can be simplified by applying the rotating wave approximation which only considers the terms which are slowly varying with time since the high-frequency terms average out to zero [5]. The quadratic terms contribute in the case NR (negative resistance parametric amplifier) for $\omega_0 \approx \omega_2 + \omega_1$, in the case UC (up-converter) for $\omega_0 \approx \omega_2 - \omega_1$, and the two degenerate parametric amplifier cases DC1 with $\omega_0 \approx 2\omega_1$, and DC2 for $\omega_0 \approx 2\omega_2$. In this case $\boldsymbol{H}_1$ can be approximated by

$$\boldsymbol{H}_1^{NR} = \gamma_{12}\left[\boldsymbol{a}_1^\dagger \boldsymbol{a}_2^\dagger e^{-j\omega_0 t} + \boldsymbol{a}_1 \boldsymbol{a}_2 e^{j\omega_0 t}\right], \quad \text{with} \;\; \gamma_{ij} = W_J \kappa_i \kappa_j = \frac{1}{2\pi^2} e_0 I_J \sqrt{Z_i Z_j}. \quad (17)$$

We compute the time evolution of $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ using the Heisenberg equation of motion [5, p. 274]

$$j\hbar \frac{d\boldsymbol{a}_{iH}}{dt} = \left[\boldsymbol{a}_{iH}, \left(\boldsymbol{H}_0 + \boldsymbol{H}_1^{NR}\right)\right]. \quad (18)$$

Using this time evolution, we can investigate the dynamics of the negative resistance type parametric amplifier. Integrating the equations of motion yields

$$\boldsymbol{a}_{1H}(t) = \mathrm{e}^{-j\omega_1 t}\left(\boldsymbol{a}_1 \cosh\gamma_{12} t + j\boldsymbol{a}_2^\dagger \sinh\gamma_{12} t\right), \quad \boldsymbol{a}_{2H}(t) = \mathrm{e}^{-j\omega_2 t}\left(\boldsymbol{a}_2 \cosh\gamma_{12} t + j\boldsymbol{a}_1^\dagger \sinh\gamma_{12} t\right). \quad (19)$$

Assuming in the initial state of the parametric amplifier a signal of amplitude w plus Gaussian noise in the signal circuit L_1C_1 and Gaussian noise only in the idler circuit L_2C_2, we have computed in [4] the time dependence of energy expectation value

$$\langle E(t)\rangle = \hbar\omega|w|^2\cosh^2\gamma_{12}t + \frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_1}{k_BT}\cosh^2\gamma_{12}t + \frac{\hbar\omega_1}{2}\coth\frac{\hbar\omega_2}{k_BT}\sinh^2\gamma_{12}t\,, \quad (20)$$

where k_B is the Boltzmann constant, T is the temperature, the first term on the right-hand side represents the amplified signal, the second term is the amplified noise of the signal circuit L_1C_1, and the third term is the amplified noise which is down-converted from the idler circuit L_2C_2 to the signal circuit.

5 Conclusions

We have shown a method for quantization of linear lossless distributed circuits. From the classical Hamilton function describing the Foster equivalent circuit of the distributed circuit, the Hamilton operator describing the circuit quantum mechanically, can be found. As an example we treated the DC–pumped Josephson parametric amplifier. In a forthcoming paper the method will be extended to multi-port structures.

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