Constraints on the temporal dispersion of passive metamaterials

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Abstract

Metamaterial applications such as cloaking, perfect lenses, and artificial permeability are restricted by the frequency dependence of the permittivity, permeability, and index of refraction. Here, causality and passivity together with integral identities for Herglotz functions are used to construct sum rules. The sum rules relate the frequency dependence of the material parameters with their high- and low-frequency values. The corresponding physical bounds determine the minimum variations of the material parameters over a frequency interval. The results are illustrated with a numerical example for artificial permeability.

1 Introduction

Metamaterials are temporally dispersive, *i.e.*, the permittivity and permeability depend on frequency. The Kramers-Kronig relations [1, 2, 3] are commonly used to model the dispersion as they relate the real and imaginary parts of the permittivity and permeability for causal material models.

In this paper, Herglotz functions and sum rules are used to derive constraints on the constitutive relations for passive material models [4]. Passivity implies that the permittivity $\epsilon(\omega)$ generates the Herglotz function $h_\epsilon(\omega) = \omega\epsilon(\omega)$ (and similarly for the permeability) [1, 4]. The sum rules relate weighted integrals of the constitutive parameter over all spectrum with the instantaneous and static response of the material model. Various sum rules are presented in [4] that constrain the dispersion of the constitutive relations. The bounds determine the minimum temporal dispersion over a frequency interval of the material parameter, *e.g.*, how close $\epsilon(\omega)$ can be to a constant ϵ_{m}.

2 Passive constitutive relations

The linear, causal, time translational invariant, continuous, isotropic, and non-magnetic constitutive relations are

$$\boldsymbol{D}(t) = \epsilon_0\epsilon_\infty \boldsymbol{E}(t) + \epsilon_0 \int_{\mathbb{R}} \chi_{\mathrm{ee}}(t-t')\boldsymbol{E}(t')\,\mathrm{d}t' \tag{1}$$

where $\chi_{\mathrm{ee}}(t) = 0$ for $t < 0$, and ϵ_0 denotes the free space permittivity. The constitutive relations are passive if

$$0 \leq \int_{-\infty}^{T} \boldsymbol{E}(t)\cdot\frac{\partial \boldsymbol{D}(t)}{\partial t}\,\mathrm{d}t = \epsilon_0\epsilon_\infty\frac{|\boldsymbol{E}(T)|^2}{2} + \epsilon_0\int_{-\infty}^{T}\int_{\mathbb{R}} \boldsymbol{E}(t)\cdot\frac{\partial}{\partial t}\big(\chi_{\mathrm{ee}}(t-t')\big)\boldsymbol{E}(t')\,\mathrm{d}t'\,\mathrm{d}t \tag{2}$$

for all times T and smooth fields $\boldsymbol{E}$ tending to zero as $t \to -\infty$.

The Fourier transform (time dependence $\mathrm{e}^{-\mathrm{i}\omega t}$) of (1) gives the frequency domain model $\boldsymbol{D}(\omega) = \epsilon_0\epsilon(\omega)\boldsymbol{E}(\omega)$ where the symbols $\boldsymbol{D}$ and $\boldsymbol{E}$ are reused to denote the electromagnetic fields as functions of the angular frequency ω. Passivity (2) restricts the permittivity ϵ such that $h_\epsilon = \omega\epsilon(\omega)$ is a Herglotz function [5, 1], *i.e.*, $h_\epsilon(\omega)$ is holomorphic and $\operatorname{Im} h_\epsilon(\omega) \geq 0$ in the upper halfplane $\operatorname{Im}\omega > 0$. The time-domain origin (1) imply the symmetry $h_\epsilon(\omega) = -h_\epsilon^*(-\omega^*)$, where a star denotes the complex conjugate. The description above is restricted to the permittivity for notational simplicity. The permeability defines an analogous Herglotz function $h_\mu(\omega) = \omega\mu(\omega)$.

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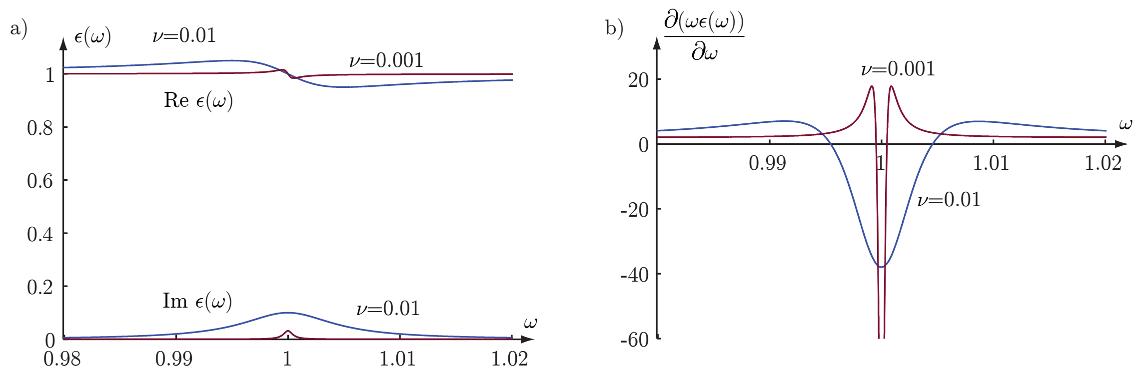

Figure 1: Illustrations of the low-loss permittivity model in (5) with $\nu = 0.01$ and $\nu = 0.001$. a) permittivity $\epsilon(\omega)$. b) derivative $\frac{\partial(\omega\epsilon)}{\partial\omega}$.

The general representation of Herglotz functions show that

$$h_\epsilon(\omega) = \omega\epsilon(\omega) = \epsilon_\infty\omega + \int_{\mathbb{R}} \left(\frac{1}{\xi-\omega} - \frac{\xi}{1+\xi^2}\right) \mathrm{d}\beta(\xi) = \epsilon_\infty\omega + \int_{\mathbb{R}} \frac{\omega}{\xi^2-\omega^2} \mathrm{d}\beta(\xi), \tag{3}$$

where $\int(1+\xi^2)^{-1}\,\mathrm{d}\beta(\xi) < \infty$, $\epsilon_\infty \geq 0$, $\mathrm{Im}\,\omega > 0$, the symmetry is used in the last equality, and $\lim_{y\to 0} \mathrm{Im}\, h(\xi + \mathrm{i}y)\,\mathrm{d}\xi = \pi\,\mathrm{d}\beta(\xi)$ if $\mathrm{Im}\, h(\xi)$ is regular [6], *cf.*, the Kramers-Kronig relations [1, 7]. The representation (3) can be used to show [4]

$$\frac{\partial h_\epsilon}{\partial\omega} = \frac{\partial(\omega\epsilon)}{\partial\omega} \geq 0 \text{ for } \omega \in [\omega_1, \omega_2] \quad \text{if } \mathrm{Im}\,\epsilon(\omega) = 0 \text{ for } \omega \in [\omega_1, \omega_2], \tag{4}$$

i.e., in frequency intervals where the material model is lossless. This restriction is not true if losses are present. This is illustrated using the Lorentz model

$$h_\epsilon(\omega) = \epsilon_\infty\omega + \frac{\omega\nu^{3/2}}{1-\omega^2-\mathrm{i}\nu\omega}, \tag{5}$$

where $\epsilon_\infty > 0$ and $\nu \geq 0$. It has $h_\epsilon(1) = \epsilon_\infty + \mathrm{i}\nu^{1/2} \approx \epsilon_\infty$ for $\nu \ll 1$ but $\frac{\partial h_\epsilon}{\partial\omega}(1) \to -\infty$ as $\nu \to 0$. The cases with $\nu = 0.01$ and $\nu = 0.001$ are illustrated in Fig. 1.

3 Sum rules and Herglotz functions

Herglotz functions, $h(z)$, are holomorphic in the upper half plane $\mathrm{Im}\, z > 0$ and map the upper half plane into itself, *i.e.*, $\mathrm{Im}\, h(z) \geq 0$, see [6, 5]. Here, we also restrict the analysis to symmetric Herglotz function $h(z) = -h^*(-z^*)$. They have at most linear growth as $z \hat{\to} \infty$ and at most a simple pole as $z \hat{\to} 0$, where $\hat{\to}$ denotes limits for $0 < \alpha \leq \arg(z) \leq \pi - \alpha$. Their asymptotic expansions are hence of the form

$$h(z) = \sum_{n=0}^{N_0} a_{2n-1} z^{2n-1} + \mathrm{o}(z^{2N_0-1}) \quad \text{as } z \hat{\to} 0 \quad \text{and } h(z) = \sum_{n=0}^{N_\infty} b_{1-2n} z^{1-2n} + \mathrm{o}(z^{1-2N_\infty}) \quad \text{as } z \hat{\to} \infty \tag{6}$$

for some $N_0 \geq 0$ and $N_\infty \geq 0$, see [6]. The expansions (6) guarantee that $\mathrm{Im}\{h(x)\}$ satisfies the integral identities

$$\lim_{\delta\to 0} \lim_{y\to 0} \frac{2}{\pi} \int_\delta^{1/\delta} \frac{\mathrm{Im}\{h(x+\mathrm{i}y)\}}{x^{2n}} \,\mathrm{d}x = \frac{2}{\pi} \int_0^\infty \frac{\mathrm{Im}\{h(x)\}}{x^{2n}} \,\mathrm{d}x = a_{2n-1} - b_{2n-1} \tag{7}$$

where $n = 1 - N_\infty, ..., N_0$ and $a_{2n-1} = b_{1-2n} = 0$ for $n < 0$, see [6] for details. Note that a simplified notation is used in this paper where the limits in (7) are dropped.

Composition of Herglotz functions is a powerful method to construct new Herglotz functions and integral identities (7). The pulse function

$$h_{\Delta}(z) = \frac{1}{\pi}\int_{|\xi|\leq\Delta}\frac{1}{\xi - z}\,\mathrm{d}\xi = \frac{1}{\pi}\ln\frac{z-\Delta}{z+\Delta} \sim \begin{cases}\mathrm{i} & \text{as } z\to 0\\ -\frac{2}{\pi}\frac{\Delta}{z} & \text{as } z\to\infty\end{cases} \tag{8}$$

is used in [4, 6] to bound the amplitude of Herglotz functions. Note that $\operatorname{Im} h_{\Delta}(x) = 1$ for $|x| < \Delta$ and $\operatorname{Im} h_{\Delta}(x) = 0$ for $|x| > \Delta$.

4 Artificial permeability

Metamaterials can produce a high-frequency permeability in otherwise non-magnetic materials [8]. For this artificial permeability, it is desired to design materials with a permeability $\operatorname{Re}\mu(\omega)$ larger than its static value μ_{s}, *e.g.*, $\mu(\omega_0) \approx \mu_{\mathrm{m}} > \mu_{\mathrm{s}}$. Here, a sum rule and its corresponding bound that only involve the static permeability is discussed. This is a complement to the results presented in [4]. The permeability generates the Herglotz function, $h_{\mu} = \omega\mu(\omega)$, with the asymptotic expansions

$$h_{\mu} = \omega\mu(\omega) \sim \begin{cases}\omega\mu_{\mathrm{s}} & \text{as } \omega\hat{\to}0\\ \omega\mu_{\infty} & \text{as } \omega\hat{\to}\infty\end{cases} \tag{9}$$

We are interested in regions $\omega \approx \omega_0$ with $\mu(\omega) \approx \mu_{\mathrm{m}}$, where $\mu_{\mathrm{m}} > \mu_{\mathrm{s}}$. It is convenient to introduce the Herglotz function

$$h_1(\omega) = \frac{-\omega_0}{h_{\mu}(\omega)} \sim \begin{cases}-\omega_0/(\omega\mu_{\mathrm{s}}) & \text{as } \omega\hat{\to}0\\ -\omega_0/(\omega\mu_{\infty}) & \text{as } \omega\hat{\to}\infty\end{cases} \tag{10}$$

Add $\omega_0/(\omega\mu_{\mathrm{m}})$ to obtain the difference

$$h_2(\omega) = h_1(\omega) + \frac{\omega_0}{\omega\mu_{\mathrm{m}}} = \frac{-\omega_0}{\omega\mu(\omega)} + \frac{\omega_0}{\omega\mu_{\mathrm{m}}} = \frac{-\omega_0}{\omega}\Big(\frac{\mu_{\mathrm{m}} - \mu(\omega)}{\mu(\omega)\mu_{\mathrm{m}}}\Big) \sim \begin{cases}-(\mu_{\mathrm{s}}^{-1} - \mu_{\mathrm{m}}^{-1})\omega_0/\omega & \text{as } \omega\hat{\to}0\\ -(\mu_{\infty}^{-1} - \mu_{\mathrm{m}}^{-1})\omega_0/\omega & \text{as } \omega\hat{\to}\infty\end{cases} \tag{11}$$

This is a Herglotz function with the property $h_3(\omega) = 0$ if $\mu(\omega) = \mu_{\mathrm{m}}$. Follow the approach in [4] and compose it with h_{Δ}, *i.e.*,

$$h_{\Delta}(h_2(\omega)) \sim \begin{cases}\frac{2\omega\Delta}{\pi\omega_0(\mu_{\mathrm{s}}^{-1} - \mu_{\mathrm{m}}^{-1})} & \text{as } \omega\hat{\to}0\\ \mathcal{O}(1) & \text{as } \omega\hat{\to}\infty\end{cases} \tag{12}$$

Use the integral identities (7) to construct the $n = 1$ sum rule

$$\int_0^{\infty}\frac{\operatorname{Im}\{h_{\Delta}(h_2(\omega))\}}{\omega^2}\,\mathrm{d}\omega = \int_0^{\infty}\frac{1}{\omega^2\pi}\arg\Big(\frac{h_2(\omega)-\Delta}{h_2(\omega)+\Delta}\Big)\,\mathrm{d}\omega = \frac{\Delta}{\omega_0(\mu_{\mathrm{s}}^{-1} - \mu_{\mathrm{m}}^{-1})} \tag{13}$$

The sum rule is illustrated in Fig. 2 for the Lorentz model $\mu(\omega) = 0.1 + 0.9/(1-\omega^2-0.01\mathrm{i}\omega)$ with $\mu_{\mathrm{m}} = 2$ and $\Delta = 0.2$. It is observed that $\mu(\omega) \approx \mu_{\mathrm{m}}$ for $\omega = \omega_0 \approx 0.73$. The Herglotz function $h_2(\omega_0) \approx 0$ and $|h_2(\omega)| \leq \Delta$ for $\omega \in [0.60, 0.87]$. This is also approximately the region with $1.4 \approx 1/(1/\mu_{\mathrm{m}} + \Delta) \leq \mu \leq 1/(1/\mu_{\mathrm{m}} - \Delta) \approx 3.3$, see the dashed lines in Fig. 2a. The corresponding composition $\operatorname{Im}\{h_{\Delta}(h_2(\omega))\}$ is concentrated around ω_0 where it is close to one. The weighted area under this curve is given by the sum rule, *i.e.*, 0.55. It is also approximately given by the width times the height. Setting them equal estimates the bandwidth to $\omega_2 - \omega_1 \approx 0.29$. Note that this is approximately the same interval where $|h_2(\omega)| \leq \Delta$.

The sum rule (13) is bounded in [4] to obtain the bound

$$\max_{\omega\in\mathcal{B}}|\mu(\omega)^{-1} - \mu_{\mathrm{m}}^{-1}| = \max_{\omega\in\mathcal{B}}\frac{|\mu(\omega) - \mu_{\mathrm{m}}|}{|\mu(\omega)|\mu_{\mathrm{m}}} \geq \frac{B}{1+B/2}\,\frac{\mu_{\mathrm{m}} - \mu_{\mathrm{s}}}{\mu_{\mathrm{s}}\mu_{\mathrm{m}}}\begin{cases}1/2 & \text{lossy case}\\ 1 & \text{lossless case.}\end{cases} \tag{14}$$

With the data from the Lorentz example, we have the fractional bandwidth $B = (\omega_2 - \omega_1)/\omega_0 \approx 0.39$ using $\Delta = 0.2$. The bound (14) gives $\max_{\omega\in\mathcal{B}}|\mu(\omega)^{-1} - \mu_{\mathrm{m}}^{-1}| \geq \{0.16, 0.08\}$ in the lossless and lossy cases, respectively.

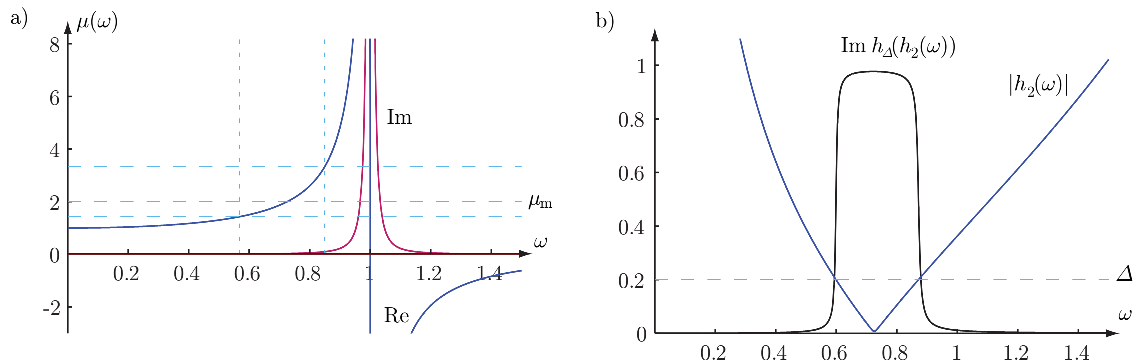

Figure 2: Illustrations of the Lorentz model $\mu(\omega) = 0.1 + 0.9/(1 - \omega^2 - 0.01\mathrm{i}\omega)$ and sum rule (13) for artificial permeability. a) the permeability $\mu(\omega)$. b) imaginary part of $h_\Delta(h_2(\omega))$ in the sum rule (13).

5 Conclusions

It is demonstrated that Herglotz functions and their associated integral identities (sum rules) provide a powerful methodology to analyze temporal dispersion of passive metamaterials. The sum rules are used to construct physical bounds that evaluated the minimum temporal dispersion of metamaterials. For example, the results for artificial permeability show that it is not possible to construct a high permeability over a large bandwidth.

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