A General Macroscopic Anisotropic Representation for Spatially Dispersive Media

Arthur D. Yaghjian[1], ***Robert A. Shore***[2], and ***Andrea Alù***[3]

[1] Research Consultant, 115 Wright Road, Concord, MA 01742 USA, a.yaghjian@comcast.net
[2] Air Force Research Laboratory, Hanscom AFB, MA 01731 USA, robert.shore@hanscom.af.mil
[3] Electrical & Computer Engineering, University of Texas, Austin, TX 78705 USA, alu@mail.utexas.edu

Abstract

It is shown that any spatially and temporally dispersive bianisotropic material, satisfying Maxwell's macroscopic equations for $\mathbf{E}$ and $\mathbf{H}$ in the Fourier transformed $(\boldsymbol{\beta},\omega)$ space, can also be represented as an anisotropic material. Thus, for many applications, magnetoelectric constitutive parameters can be avoided at the macroscopic level.

1 Introduction

Nearly all metamaterials, unlike most natural materials, are characterized by strong spatial dispersion, as well as temporal (frequency) dispersion, and thus with the increasing interest in metamaterials during the past several years, spatial dispersion has also gained increasing importance and attention. Although the detailed fields of a classical model of materials/metamaterials require an analysis based on Maxwell's microscopic equations, enormous simplification results if the material can be represented to a good approximation by Maxwell's macroscopic equations with macroscopic sources and fields related by constitutive parameters. Spatial and temporal dispersion in these macroscopic equations are conveniently represented by four-fold Fourier transforms of the macroscopic Maxwellian fields and sources in the space-time $(\mathbf{r},t)$ domain that yield macroscopic Maxwell's equations in the $(\boldsymbol{\beta},\omega)$ Fourier domain. Each of the source and field vectors in the macroscopic $(\boldsymbol{\beta},\omega)$ domain can then be looked upon as having $\exp(i\boldsymbol{\beta}\cdot\mathbf{r}-i\omega t)$ dependence for any real value of the three components of the propagation vector $\boldsymbol{\beta}$ and of the frequency ω. The most general spatially and temporally dispersive, translationally invariant, linear medium is usually characterized by four bianisotropic constitutive parameters, the permittivity and permeability dyadics and the two magnetoelectric dyadics. In this paper we show that these four dyadics in the bianisotropic spatially and temporally dispersive representation can be reduced to two dyadics, a permittivity dyadic and a permeability dyadic, in an anisotropic spatially and temporally dispersive representation. Moreover, this reduction is accomplished without changing the conventional definitions of the macroscopic electric and magnetic fields or the macroscopic electric and magnetic polarization densities.

2 Maxwell's Macroscopic Equations in the Space-Time $(\mathbf{r},t)$ Domain

We begin by assuming a polarizable material that can be described electromagnetically by the following macroscopic Maxwell differential equations for a polarizable continuum in the space-time $(\mathbf{r},t)$ domain [1, ch. 1]

$$\nabla\times\boldsymbol{\mathcal{E}}+\mu_0\frac{\partial\boldsymbol{\mathcal{H}}}{\partial t}+\mu_0\frac{\partial\boldsymbol{\mathcal{M}}_{\rm ind}}{\partial t}=-\mu_0\frac{\partial\boldsymbol{\mathcal{M}}_{\rm a}}{\partial t},\quad \nabla\cdot\boldsymbol{\mathcal{H}}+\nabla\cdot\boldsymbol{\mathcal{M}}_{\rm ind}=-\nabla\cdot\boldsymbol{\mathcal{M}}_{\rm a} \tag{1a}$$

$$\nabla\times\boldsymbol{\mathcal{H}}-\epsilon_0\frac{\partial\boldsymbol{\mathcal{E}}}{\partial t}-\frac{\partial\boldsymbol{\mathcal{P}}_{\rm ind}}{\partial t}-\boldsymbol{\mathcal{J}}_{\rm ind}=\boldsymbol{\mathcal{J}}_{\rm a}+\frac{\partial\boldsymbol{\mathcal{P}}_{\rm a}}{\partial t},\quad \epsilon_0\nabla\cdot\boldsymbol{\mathcal{E}}+\nabla\cdot\boldsymbol{\mathcal{P}}_{\rm ind}-\rho_{\rm ind}=\rho_{\rm a}-\nabla\cdot\boldsymbol{\mathcal{P}}_{\rm a} \tag{1b}$$

in which the current-charge and polarization densities have been divided into induced and applied source densities. These space-time Maxwell equations in (1) cannot be solved without relating the induced source densities to the fields $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{H}}$, which will be taken as the primary fields with the electric and magnetic dipoles comprising $\boldsymbol{\mathcal{P}}=\boldsymbol{\mathcal{P}}_{\rm a}+\boldsymbol{\mathcal{P}}_{\rm ind}$ and $\boldsymbol{\mathcal{M}}=\boldsymbol{\mathcal{M}}_{\rm a}+\boldsymbol{\mathcal{M}}_{\rm ind}$ given in terms of electric and magnetic charge separation, respectively. These space-time equations require four-fold convolution integrals to express reasonable linear relationships between the induced source densities and the primary fields for media exhibiting both spatial and temporal dispersion [2, sec. 103]. This makes it practically impossible to solve these Maxwell equations directly in the space-time domain. Therefore, in the following section, these space-time equations are Fourier transformed to convert them to simpler plane-wave equations for the transformed fields as functions of the real-valued Fourier transform variables, $\boldsymbol{\beta}$ and ω, corresponding to $\mathbf{r}$ and t.

3 Maxwell's Macroscopic Equations in the Spectral $(\boldsymbol{\beta},\omega)$ Domain

Since each of the fields and source densities in (1) are functions of $(\mathbf{r},t)$, they can be expanded in four-dimensional Fourier transforms. Taking the electric field as an example, we have

978-1-4244-6051-9/11/$26.00 ©2011 IEEE

$$\boldsymbol{\mathcal{E}}(\mathbf{r},t) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathbf{E}(\boldsymbol{\beta},\omega)e^{i(\boldsymbol{\beta}\cdot\mathbf{r}-\omega t)}d^3\boldsymbol{\beta}d\omega, \quad \mathbf{E}(\boldsymbol{\beta},\omega) = \frac{1}{(2\pi)^4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \boldsymbol{\mathcal{E}}(\mathbf{r},t)e^{-i(\boldsymbol{\beta}\cdot\mathbf{r}-\omega t)}d^3\mathbf{r}dt \quad (2)$$

with $d^3\boldsymbol{\beta} \equiv d\beta_x d\beta_y d\beta_z$ and $d^3\mathbf{r} \equiv dxdydz$. Like the components of the vector $\mathbf{r}$ and the time t, the three components of the Fourier transform propagation vector $\boldsymbol{\beta}$ and the frequency ω are real and span the full real line from $-\infty$ to $+\infty$. Inserting the integral for $\boldsymbol{\mathcal{E}}(\mathbf{r},t)$ from (2) and the similar integrals for all the other fields and source densities into (1), then taking the Fourier transforms, convert Maxwell's macroscopic equations in the space-time $(\mathbf{r},t)$ domain to Maxwell's macroscopic equations in the spectral $(\boldsymbol{\beta},\omega)$ domain, namely

$$\boldsymbol{\beta}\times\mathbf{E} - \omega\mu_0\mathbf{H} - \omega\mu_0\mathbf{M}_{\text{ind}} = \omega\mu_0\mathbf{M}_{\text{a}} \quad (3a)$$

$$\boldsymbol{\beta}\times\mathbf{H} + \omega\epsilon_0\mathbf{E} + \omega\mathbf{P}_{\text{ind}} = -\omega\mathbf{P}_{\text{a}}\,. \quad (3b)$$

For $\omega \neq 0$, we require only the two Maxwell curl equations in the $(\boldsymbol{\beta},\omega)$ domain because the two divergence equations follow from the continuity equations. In the two Maxwell equations in (3), the effective applied and induced electric polarization densities $\mathbf{P}_{\text{a}} + i\mathbf{J}_{\text{a}}/\omega$ and $\mathbf{P}_{\text{ind}} + i\mathbf{J}_{\text{ind}}/\omega$ are denoted simply by $\mathbf{P}_{\text{a}}$ and $\mathbf{P}_{\text{ind}}$, respectively. The total effective electric polarization density and effective electric displacement are given by $\mathbf{P} = \mathbf{P}_{\text{a}} + \mathbf{P}_{\text{ind}}$ and $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, respectively. Similarly, for the magnetization density and magnetic induction, we have $\mathbf{M} = \mathbf{M}_{\text{a}} + \mathbf{M}_{\text{ind}}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, respectively. It is emphasized that the applied sources $(\mathbf{M}_{\text{a}}, \mathbf{P}_{\text{a}})$ are required in (3) in order to allow arbitrary real values of both ω and the three components of $\boldsymbol{\beta}$ in the subsequent linear formulations given in (5) or (6) or (11) [3], [4], [5]. If the applied sources are zero, the resultant source-free equations in (5) or (6) or (11) are satisfied only by the source-free traveling waves with $\boldsymbol{\beta}$ equal to a function (sometimes a multivalued function) of ω, that is, $\boldsymbol{\beta}(\omega)$ is an eigenvalue related to ω by a "dispersion equation" [4], [6, sec. 2.3].

3.1 Bianisotropic Constitutive Relations

To obtain the solution for $\mathbf{E}$ and $\mathbf{H}$ in (3) for given applied polarization densities $\mathbf{M}_{\text{a}}$ and $\mathbf{P}_{\text{a}}$, the induced polarization densities $\mathbf{M}_{\text{ind}}$ and $\mathbf{P}_{\text{ind}}$ must be given in terms of $\mathbf{E}$ and $\mathbf{H}$. Bianisotropic linear relationships between the induced polarization densities and the $\mathbf{E}$ and $\mathbf{H}$ fields are given by

$$\mathbf{P}_{\text{ind}}(\boldsymbol{\beta},\omega) = \epsilon_0\overline{\boldsymbol{\chi}}_e(\boldsymbol{\beta},\omega)\cdot\mathbf{E}(\boldsymbol{\beta},\omega) + \overline{\boldsymbol{\tau}}(\boldsymbol{\beta},\omega)\cdot\mathbf{H}(\boldsymbol{\beta},\omega)/c \quad (4a)$$

$$\mathbf{M}_{\text{ind}}(\boldsymbol{\beta},\omega) = \overline{\boldsymbol{\chi}}_m(\boldsymbol{\beta},\omega)\cdot\mathbf{H}(\boldsymbol{\beta},\omega) + \overline{\boldsymbol{\nu}}(\boldsymbol{\beta},\omega)\cdot\mathbf{E}(\boldsymbol{\beta},\omega)/Z_0 \quad (4b)$$

where $\overline{\boldsymbol{\chi}}_e$ and $\overline{\boldsymbol{\chi}}_m$ are the electric and magnetic susceptibility dyadics, $(\overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\nu}})$ are the magnetoelectric dyadics, and $Z_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. The constitutive dyadics in (4) relate the induced polarization densities to the total plane-wave fields at the single spatial frequency $\boldsymbol{\beta}$ and single temporal frequency ω. In general, these constitutive parameters are functions of $\boldsymbol{\beta}$ as well as ω and thus the material exhibits spatial dispersion as well as temporal (frequency) dispersion, although the material is time invariant and "homogeneous" in the sense that the macroscopic electromagnetic properties of the material do not change with time and with a translation of the coordinate system of the observer. Unless the material is also isotropic, the components of the dyadic constitutive parameters will change with rotation of the coordinate system. Substitution of $\mathbf{P}_{\text{ind}}$ and $\mathbf{M}_{\text{ind}}$ from (4) into (3) produces the $(\mathbf{E},\mathbf{H})$ form of the bianisotropic, spatially dispersive macroscopic Maxwell equations

$$\boldsymbol{\beta}\times\mathbf{E} - \omega\overline{\boldsymbol{\mu}}_{\text{b}}\cdot\mathbf{H} - \omega\overline{\boldsymbol{\nu}}\cdot\mathbf{E}/c = \omega\mu_0\mathbf{M}_{\text{a}}, \quad \overline{\boldsymbol{\mu}}_{\text{b}} = \mu_0(\overline{\boldsymbol{\chi}}_m + \overline{\mathbf{I}}) \quad (5a)$$

$$\boldsymbol{\beta}\times\mathbf{H} + \omega\overline{\boldsymbol{\epsilon}}_{\text{b}}\cdot\mathbf{E} + \omega\overline{\boldsymbol{\tau}}\cdot\mathbf{H}/c = -\omega\mathbf{P}_{\text{a}}, \quad \overline{\boldsymbol{\epsilon}}_{\text{b}} = \epsilon_0(\overline{\boldsymbol{\chi}}_e + \overline{\mathbf{I}})\,. \quad (5b)$$

3.2 Reduction of Bianisotropic Equations to Anisotropic Equations

It will now be shown that the four-dyadic, bianisotropic, spatially dispersive, macroscopic Maxwell equations in (5) can be reduced to anisotropic, spatially dispersive, macroscopic Maxwell equations involving two dyadics, $\overline{\boldsymbol{\epsilon}}(\boldsymbol{\beta},\omega)$ and $\overline{\boldsymbol{\mu}}(\boldsymbol{\beta},\omega)$, thus rendering bianisotropy a formally unnecessary macroscopic description for spatially dispersive materials. To do this, simply solve for $\mathbf{E}$ in (5b) and $\mathbf{H}$ in (5a), then insert them into (5a) and (5b), respectively, to get

$$\boldsymbol{\beta}\times\mathbf{E} - \omega\overline{\boldsymbol{\mu}}\cdot\mathbf{H} = \omega\mu_0\,\mathbf{M}'_{\text{a}} \quad (6a)$$

$$\boldsymbol{\beta}\times\mathbf{H} + \omega\overline{\boldsymbol{\epsilon}}\cdot\mathbf{E} = -\omega\mathbf{P}'_{\text{a}} \quad (6b)$$

in which

$$\overline{\boldsymbol{\mu}} = \overline{\boldsymbol{\mu}}_{\rm b} - \overline{\boldsymbol{\nu}} \cdot \overline{\boldsymbol{\epsilon}}_{\rm b}^{-1} \cdot \left(\overline{\boldsymbol{\tau}}/c + \overline{\boldsymbol{\beta}}/\omega\right)/c, \quad \mathbf{M}'_{\rm a} = \mathbf{M}_{\rm a} - \overline{\boldsymbol{\nu}} \cdot \overline{\boldsymbol{\epsilon}}_{\rm b}^{-1} \cdot \mathbf{P}_{\rm a}/Z_0 = \mathbf{M}_{\rm a} + \mathbf{M}_{\rm ba} \tag{7a}$$

$$\overline{\boldsymbol{\epsilon}} = \overline{\boldsymbol{\epsilon}}_{\rm b} - \overline{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{\mu}}_{\rm b}^{-1} \cdot \left(\overline{\boldsymbol{\nu}}/c - \overline{\boldsymbol{\beta}}/\omega\right)/c, \quad \mathbf{P}'_{\rm a} = \mathbf{P}_{\rm a} - \mu_0 \overline{\boldsymbol{\tau}} \cdot \overline{\boldsymbol{\mu}}_{\rm b}^{-1} \cdot \mathbf{M}_{\rm a}/c = \mathbf{P}_{\rm a} + \mathbf{P}_{\rm ba}\,. \tag{7b}$$

(The antisymmetric dyadic $\overline{\boldsymbol{\beta}}$ is used to replace $\boldsymbol{\beta}\times$; that is, $\overline{\boldsymbol{\beta}} \cdot \mathbf{V} = \boldsymbol{\beta} \times \mathbf{V}$ for any vector $\mathbf{V}$.) Equations (6) imply that *any spatially dispersive bianisotropic material satisfying Maxwell's macroscopic equations for* $\mathbf{E}$ *and* $\mathbf{H}$ *in* $(\boldsymbol{\beta}, \omega)$ *space* ($\omega \neq 0$) *can also be represented as an anisotropic material. A macroscopic bianisotropic representation is not required for spatially dispersive media.* Moreover, because the $\mathbf{E}$ and $\mathbf{H}$ fields in (5) and (6) are the same, the scattered (reflected and transmitted) fields from a finite volume of the material without embedded applied sources are the same for the bianisotropic and anisotropic representations (assuming (6) does not have more solutions than (5)). If we retain the usual definitions of $\mathbf{D}$ and $\mathbf{B}$ in terms of $\mathbf{E}$, $\mathbf{H}$, $\mathbf{P}$, and $\mathbf{M}$, then

$$\mathbf{D} = \overline{\boldsymbol{\epsilon}} \cdot \mathbf{E} + \mathbf{P}'_{\rm a} = \epsilon_0 \mathbf{E} + \mathbf{P}, \quad \mathbf{P}'_{\rm ind} = \mathbf{P} - \mathbf{P}'_{\rm a} = (\overline{\boldsymbol{\epsilon}} - \epsilon_0 \overline{\mathbf{I}}) \cdot \mathbf{E} = \epsilon_0 \overline{\boldsymbol{\chi}}_{\rm E} \cdot \mathbf{E} \tag{8a}$$

$$\mathbf{B} = \overline{\boldsymbol{\mu}} \cdot \mathbf{H} + \mu_0 \mathbf{M}'_{\rm a} = \mu_0 (\mathbf{H} + \mathbf{M}), \quad \mathbf{M}'_{\rm ind} = \mathbf{M} - \mathbf{M}'_{\rm a} = (\overline{\boldsymbol{\mu}}/\mu_0 - \overline{\mathbf{I}}) \cdot \mathbf{H} = \overline{\boldsymbol{\chi}}_{\rm M} \cdot \mathbf{H}\,. \tag{8b}$$

The modified "induced" polarizations $\mathbf{P}'_{\rm ind}$ and $\mathbf{M}'_{\rm ind}$ are introduced merely as symbols for $\epsilon_0 \overline{\boldsymbol{\chi}}_{\rm E} \cdot \mathbf{E}$ and $\overline{\boldsymbol{\chi}}_{\rm M} \cdot \mathbf{H}$, and are related to the original induced polarizations by $\mathbf{P}'_{\rm ind} = \mathbf{P}_{\rm ind} - \mathbf{P}_{\rm ba}$ and $\mathbf{M}'_{\rm ind} = \mathbf{M}_{\rm ind} - \mathbf{M}_{\rm ba}$. Neither $(\overline{\boldsymbol{\chi}}_{\rm E}, \overline{\boldsymbol{\chi}}_{\rm M})$ nor $(\mathbf{P}'_{\rm ind}, \mathbf{M}'_{\rm ind})$ need be introduced in working with the anisotropic equations in (6). In the anisotropic equations (6), the original bianisotropic contributions are replaced by the applied source terms $\mathbf{P}_{\rm ba}$ and $\mathbf{M}_{\rm ba}$ defined in (7) and included in the effective applied polarization source densities $\mathbf{P}'_{\rm a}$ and $\mathbf{M}'_{\rm a}$ on the right-hand sides of (6), where these effective applied polarization densities are modifications of the original applied polarization densities in (5). In many practical applications, however, one is most interested in solving for the traveling waves in source-free media, that is, with the applied sources $\mathbf{P}_{\rm a}$ and $\mathbf{M}_{\rm a}$ equal to zero, and thus with $\mathbf{P}'_{\rm a}$ and $\mathbf{M}'_{\rm a}$ equal to zero. In this circumstance, the simplicity of the anisotropic representation in (6) may be particularly advantageous. It is emphasized that in deriving the anisotropic Maxwellian equations in (6) for spatially dispersive materials, neither the definitions of the macroscopic $\mathbf{E}$ and $\mathbf{H}$ fields, nor of $\mathbf{P}$, $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{B}$ have been changed from their conventional definitions. Still, a bianisotropic representation may be preferred for materials that are spatially nondispersive in the frequency range of interest, since in this case the anisotropic representation would introduce an artificial spatial dispersion. That is, suppose one is interested in the source-free fields of a material in a range of $[\boldsymbol{\beta}(\omega), \omega]$ where the bianisotropic parameters $(\overline{\boldsymbol{\epsilon}}_{\rm b}, \overline{\boldsymbol{\mu}}_{\rm b}, \overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\nu}})$ in (5) are independent of $\boldsymbol{\beta}(\omega)$. One can still recast the equations in the anisotropic form of (6), but then the $\overline{\boldsymbol{\epsilon}}$ and $\overline{\boldsymbol{\mu}}$ in (6) will be functions of $\boldsymbol{\beta}$ (as well as ω) even though the original $(\overline{\boldsymbol{\epsilon}}_{\rm b}, \overline{\boldsymbol{\mu}}_{\rm b}, \overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\nu}})$ in (5) are not functions of $\boldsymbol{\beta}$. Thus, the spatially nondispersive bianisotropic medium becomes spatially dispersive if transformed to an anisotropic medium. In such cases, one has the choice of a four-dyadic spatially nondispersive bianisotropic medium or a two-dyadic spatially dispersive anisotropic medium. The microscopic interaction of the dipole moments in metamaterial arrays is fundamentally bianisotropic because the fields of either the electric or magnetic dipole moment of one inclusion generally induces both electric and magnetic dipole moments in all the other inclusions. Thus, it is somewhat natural to maintain this bianisotropic representation on a macroscopic level as well if one accepts the extra burden of dealing with magnetoelectric parameters. These bianisotropic constitutive parameters can maintain appealing properties, such as the continuum passivity conditions in bandgap regions where $\boldsymbol{\beta}$ is complex and the anisotropic constitutive parameters can disobey the passivity conditions [7], [8], although, as mentioned above, both the anisotropic and bianisotropic constitutive parameters produce the same fields and thus the same external reflection and transmission coefficients.

3.3 Landau-Lifshitz Single-Polarization-Dyadic Formulation for Spatial Dispersion

In the previous section, it was shown that Maxwell's macroscopic equations in spatially dispersive materials involving four bianisotropic constitutive dyadics, for example $(\overline{\boldsymbol{\chi}}_e, \overline{\boldsymbol{\chi}}_m, \overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\nu}})$ in (4) or $(\overline{\boldsymbol{\epsilon}}_{\rm b}, \overline{\boldsymbol{\mu}}_{\rm b}, \overline{\boldsymbol{\tau}}, \overline{\boldsymbol{\nu}})$ in (5) , could be reduced to a macroscopic anisotropic representation involving just two dyadics, namely $(\overline{\boldsymbol{\mu}}, \overline{\boldsymbol{\epsilon}})$ in (6). It can also be shown that by combining electric and magnetic polarization source densities into one vector, Maxwell's equations in spatially dispersive media can be re-expressed in a form involving just a single polarization dyadic [2, sec. 103], [3], [4], [9]. To derive these single-polarization-dyadic equations, rewrite the $(\boldsymbol{\beta}, \omega)$ domain equations in (3) as

$$\boldsymbol{\beta} \times \mathbf{E} - \omega \mathbf{B} = 0, \quad \boldsymbol{\beta} \times \mathbf{B} + \omega \mu_0 \epsilon_0 \mathbf{E} + \omega \mu_0 \mathbf{P}^g_{\rm ind} = -\omega \mu_0 \mathbf{P}^g_{\rm a} \tag{9}$$

where the "generalized" induced and applied polarizations are defined as $\mathbf{P}^g_{\rm ind} = \mathbf{P}_{\rm ind} - \boldsymbol{\beta} \times \mathbf{M}_{\rm ind}/\omega$ and $\mathbf{P}^g_{\rm a} = \mathbf{P}_{\rm a} - \boldsymbol{\beta} \times \mathbf{M}_{\rm a}/\omega$ with $\mathbf{P}^g = \mathbf{P}^g_{\rm ind} + \mathbf{P}^g_{\rm a}$. Assuming linear constitutive relations for $\mathbf{P}_{\rm ind}$ and $\mathbf{M}_{\rm ind}$ in terms of $\mathbf{E}$ and $\mathbf{B}$, the vector $\mathbf{B}$ in these relations can be expressed as $\boldsymbol{\beta} \times \mathbf{E}/\omega$ from (9). Thus, the induced generalized polarization $\mathbf{P}^g_{\rm ind}$ (and displacement vector $\mathbf{D}_g$) can be linearly related to $\mathbf{E}$ by a generalized susceptibility dyadic $\overline{\boldsymbol{\chi}}_g$ such that

$$\mathbf{P}^g_{\rm ind} = \epsilon_0 \overline{\boldsymbol{\chi}}_g \cdot \mathbf{E}, \qquad \mathbf{D}_g = \overline{\boldsymbol{\epsilon}}_g \cdot \mathbf{E} + \mathbf{P}^g_{\rm a} \tag{10}$$

with $\overline{\boldsymbol{\epsilon}}_g = \epsilon_0(\overline{\mathbf{I}} + \overline{\boldsymbol{\chi}}_g)$. Consequently, the spatially dispersive Maxwell equations in (9) reduce to simply

$$\boldsymbol{\beta} \times \mathbf{E} - \omega \mathbf{B} = 0 \tag{11a}$$

$$\boldsymbol{\beta} \times \mathbf{B} + \omega\mu_0 \overline{\boldsymbol{\epsilon}}_g \cdot \mathbf{E} = -\omega\mu_0 \mathbf{P}^g_{\rm a} \tag{11b}$$

a set of equations with two field unknowns, $\mathbf{E}$ and $\mathbf{B}$, and one constitutive dyadic $\overline{\boldsymbol{\epsilon}}_g$. The single-polarization-dyadic macroscopic formulation in (11) combines $\mathbf{P}$ and $\mathbf{M}$ into a single polarization vector $\mathbf{P}^g$. This combination of both the electric dipole moments and the magnetic dipole moments into a single polarization density stands in contrast to the formulations found in most of the literature dating back to the 19th century that separate electric dipole moments from the magnetic dipole moments. Moreover, the initial appeal of the formulation in (11) requiring a single constitutive dyadic $\overline{\boldsymbol{\epsilon}}_g$ is mitigated if one tries to use the equations in (11) for practical applications. For example, the dyadic $\overline{\boldsymbol{\epsilon}}_g$ does not reduce to a scalar even if the media is isotropic — thereby requiring in reality no fewer unknowns than the anisotropic formulation in (6). In addition, solving for the traveling waves supported by arrays often requires determining the fields in one unit cell produced by the electric and magnetic dipole moments of all the other unit cells. Thus, the anisotropic formulation in (6) that separates the contributions from the electric and magnetic dipole moments is well-suited for these solutions, whereas the compact single-polarization-dyadic formulation in (11) that does not distinguish between the electric and magnetic dipole contributions proves inconvenient unless it is effectively converted to the anisotropic formulation in (6). In particular, we have found that the anisotropic formulation in (6) is more convenient than either the bianisotropic formulation in (5) and the single-polarization-dyadic formulation in (11) for the macroscopic characterization of source-free traveling waves on arrays of magnetodielectric spheres [10], [11]. Nonetheless, the simplicity and elegance that the single-polarization-dyadic provides for homogenization theories is certainly appealing [2]–[4], [9].

In concluding, we mention that none of the macroscopic representations (bianisotropic, anisotropic, single-polarization) necessarily accurately model the behavior of metamaterial arrays unless the individual and combined power flow in the higher-order Floquet modes of the array are a small fraction of the power flow in the primary Floquet mode — a condition that is generally met if the phase of the fields varies less than about a radian over the unit cell.

4 References

1. J.A. Stratton. *Electromagnetic Theory*. McGraw-Hill, New York, 1941.

2. L.D. Landau, E.M. Lifshitz, and L.P. Pitaevskii. *Electrodynamics of Continuous Media*. B-H, Oxford, 1984.

3. V.P. Silin and A.A. Rukhadze. *Electromagnetic Properties of Plasmas and Plasma-Like Media*. Gosatomizdat, Moscow, 1961 [in Russian].

4. V.M. Agranovich and V.L. Ginzburg. *Spatial Dispersion in Crystal Optics and the Theory of Excitons*. Wiley-Interscience, New York, 1966; also see 2nd Edition, Springer-Verlag, New York, 1984.

5. C. Fietz and G. Shvets. Current-driven metamaterial homogenization. *Physica B*, 10:1016, 2010.

6. J.A. Kong. *Electromagnetic Wave Theory*. Wiley Press, New York, 1986.

7. A. Alù. First-principle homogenization theory for periodic metamaterial arrays. http://arxiv.org/abs/1012.1351.

8. A. Alù. Restoring the physical meaning of metamaterial constitutive parameters. *PRB*, 83:081102(R), 2011.

9. M.G. Silveirinha. Nonlocal homogenization theory of structured materials. In F. Capolino, editor, *Metamaterials Handbook: Theory and Phenomena of Metamaterials*, chapter 10. CRC Press, Boca Raton, FL, 1989.

10. R.A. Shore and A.D. Yaghjian. Traveling waves on two- and three-dimensional periodic arrays of lossless scatterers. *Radio Science*, 42, RS6S21, 2007; Correction. 43, RS2S99, 2008.

11. R.A. Shore and A.D. Yaghjian. *Complex Waves on 1D, 2D, and 3D Periodic Arrays of Lossy and Lossless Magnetodielectric Spheres*. AFRL In-House Report AFRL-RY-HS-TR-2010-0019, 2010.