Spectral properties of the quarter-plane problem using the Wiener-Hopf method

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Abstract

This paper reviews the two dimensional spectral formulation of the quarter-plane problem using the Wiener-Hopf method. Properties of this representation are investigated and an approximate method for the solution is proposed using the Fredholm factorization method together with recursive equations.

1. Introduction

The diffracted ray contributions from a vertex of a quarter plane constitute a challenging and important problem. In particular the 2004 URSI Commission proposed a prize which has the goal of a rapid determination of the diffraction coefficients by a quarter-plane. In fact the existing solution (based on the observation that the quarter plane is a degenerate elliptic cone) yields an exact series representation that shows a very poor convergence rate, in particular when the observation point is far from the tip of the quarter-plane. However, alternative approaches based on spectral formulations do not produced exact solutions up to now. In particular the well-known Radlow solution, which is based on the Wiener-Hopf factorization of an analytical function of two variables, is wrong. Albani [1] proved the incorrectness of the Radlow's demonstration. An approximated technique of factorization has been proposed in [2]. This technique is based on the solution of Fredholm integral equations of second kind. The solutions of the Fredholm equations can be obtained through different methods. Since the approximate numerical solution provides a representation of the Wiener-Hopf unknowns only in certain regions of the two-dimensional spectral domain, a procedure of analytical continuation is required to extend the domain of the solution. The analytical continuation of numerical results is an old and cumbersome problem of applied mathematics that can be approached in various ways. In this work we propose recursive equations. In particular to get these equations we introduce two angular complex planes and take into account the properties of the Wiener-Hopf unknowns in these planes.

2. The Wiener-Hopf equation

For the sake of simplicity in this paper we consider only the case of scalar diffraction in presence of a soft quarter of plane defined by $z=0$, $x \geq 0$, $y \geq 0$, see Fig. 1. The source of the problem is constituted by the incident plane wave:

$$\psi^i(x,y,z) = e^{-j\eta_o x - j\xi_o x - j\alpha_o z} \tag{1}$$

where k is the propagation constant in the free space and θ_o and φ_o are the azimuthal and zenithal angles of the incident plane wave thus:

$$\eta_o = k\sin\theta_o\cos\varphi_o, \quad \xi_o = k\sin\vartheta_o\sin\varphi_o, \quad \alpha_o = \sqrt{k^2-\eta_o^2-\xi_o^2} = k\cos\vartheta_o,$$

The total field $\psi(x,y,z)$ must solve the equation:

$$\nabla^2\psi + k^2\psi = 0 \tag{2}$$

with the boundary equation: $\psi(x,y,0)=0 \qquad x \geq 0, y \geq 0$.

The scattered field $\psi^s(x,y,z)$ satisfies the equation:

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$$\nabla^2\psi^s + k^2\psi^s = j(x,y)\delta(z) \tag{3}$$

where $j(x,y)$ is the equivalent source induced on the quarter of plane.

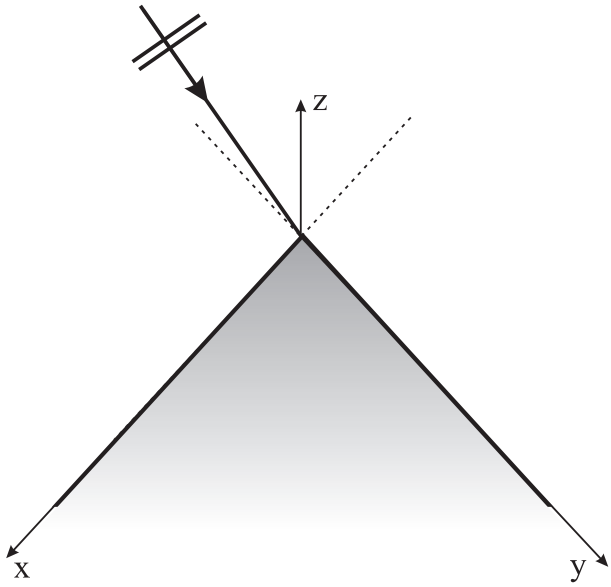

Fig. 1: Geometry of the quarter-plane problem

To understand the mathematical meaning of $j(x,y)$ we apply the operator $\int_{0_-}^{0_+} ...dz$ to the above equation; it yields:

$$\left.\frac{\partial\psi(x,y,z)}{\partial z}\right|_{z=0_+} - \left.\frac{\partial\psi(x,y,z)}{\partial z}\right|_{z=0_-} = j(x,y)$$

Taking into account the relationship:

$$(\nabla^2 + k^2)\frac{e^{-jk\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}} == -4\pi\delta(x-x')\delta(y-y')\delta(z)$$

the following Green representation holds:

$$\psi(x,y,z) = \psi^i(x,y,z) - \frac{1}{4\pi}\int_0^\infty\int_0^\infty \frac{e^{-jk\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{(x-x')^2+(y-y')^2+z^2}} j(x',y')dx'dy'$$

In the region $\{ z=0, x\geq 0, y\geq 0 \}$ $\psi(x,y,z)=0$ and the two-dimensional W-H equation holds:

$$\int_0^\infty\int_0^\infty g(x-x',y-y')j(x',y')dx'dy' = \psi^i(x,y,0) \qquad x\geq 0, y\geq 0 \tag{4}$$

where $g(x,y) = \frac{1}{4\pi}\frac{e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}$

The double Fourier Transform of $g(x,y)$ is given by:

$$G(\eta,\xi)=\int_{-\infty}^{\infty}\int_{\infty}^{\infty}g(x,y)e^{j\eta x}e^{j\xi y}dxdy=\frac{1}{4\pi}\int_{-\infty}^{\infty}\int_{\infty}^{\infty}\frac{e^{-jk\sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}}e^{j\eta x}e^{j\xi y}dxdy=-j\frac{1}{2}\frac{1}{\sqrt{k^2-\eta^2-\xi^2}} \quad (5)$$

It yields the following two-dimensional Wiener-Hopf equation in the spectral domain:

$$G(\eta,\xi)F_+(\eta,\xi)=F_+^i(\eta,\xi)+F_-(\eta,\xi) \quad (6)$$

where:

$$F_+(\eta,\xi)=J_{++}(\eta,\xi)=\int_0^{\infty}\int_0^{\infty}j(x,y)e^{j\eta x}e^{j\xi y}dxdy$$

$$F_+^i(\eta,\xi)=\Psi_{++}^i(\eta,\xi)=\int_0^{\infty}\int_0^{\infty}e^{-j\eta_o x-j\xi_o x}e^{j\eta x-j\xi x}dxdy=-\frac{1}{\eta-\eta_o}\frac{1}{\xi-\xi_o}$$

$$F_-(\eta,\xi)=V_{--}(\eta,\xi)+Y_{-+}(\eta,\xi)+Z_{+-}(\eta,\xi)$$

In the above definitions, the subscripts refer to the half-planes of regularity located in η and ξ respectively. For instance the subscript $+-$ means that the considered function is (independently on the value of the complex parameter ξ) regular in an upper half-plane of the complex variable η and (independently on the value of the complex parameter η) regular in a lower half-plane of the complex variable ξ.
The inverse Fourier transform of a function with subscript $++$, produces a function that in the spatial domain is vanishing as $x<0$ and $y<0$. Similarly it happens for the other subscripts. For instance the subscript $-+$ produces a function that in the spatial domain is vanishing as $x>0$ and $y<0$.

3. Fredholm equation of the problem

In the framework of the Wiener-Hopf technique the solution of the convolutional integral equation (4) is reduced to the two-dimensional factorization of the kernel (5). The factorization is obtained resorting to the solution of the following integral equations [2]:

$$G(\eta,\xi)F_+(\eta,\xi)+\frac{1}{(2\pi j)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{G(\eta',\xi')-G(\eta,\xi)}{(\eta'-\eta)(\xi'-\xi)}F_+(\eta',\xi')d\eta'd\xi'=-\frac{1}{\eta-\eta_o}\frac{1}{\xi-\xi_o} \quad (7)$$

Equation (2) is a two-dimensional Fredholm integral equation of the second kind that can be solved by using several well known in literature techniques.
As in the one dimensional case, the accuracy of the numerical solutions of (7) considerably increases with the deformation of the contour path. For example, the real axis of the $\eta-plane$ ($\xi-plane$) is warped into the straight line λ_η (λ_ξ) that joins the points $-jk$ and $+jk$. Besides, we introduce the complex planes w_η, w_ξ (8) in order to facilitate function-theoretic manipulations [2]:

$$\eta=-k\cos w_\eta,\quad \xi=-k\cos w_\xi \quad (8)$$

In these planes the straight lines λ_η and λ_ξ are represented by the vertical lines: $w_\eta=-\frac{\pi}{2}+ju_\eta,\ w_\xi=-\frac{\pi}{2}+ju_\xi$ where u_η and u_ξ are real. With these new variables equation (7) becomes:

$$\hat{G}(u_\eta,u_\xi)\hat{F}_+(u_\eta,u_\xi)+\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{M(u_\eta,u_\xi,u_\eta',u_\xi')\operatorname{ch}u_\eta'\operatorname{ch}u_\xi'}{(\operatorname{sh}u_\eta'-\operatorname{sh}u_\eta)(\operatorname{sh}u_\xi'-\operatorname{sh}u_\xi)}\hat{F}_+(u_\eta',u_\xi')du_\eta'du_\xi'=-\frac{1}{j\operatorname{sh}u_\eta-\eta_o}\frac{1}{j\operatorname{sh}u_\xi-\xi_o} \quad (9)$$

where $\hat{G}(u_\eta,u_\xi)=G(\eta,\xi)$, $\hat{F}_+(u_\eta,u_\xi)=F_+(\eta,\xi)$; $M(u_\eta,u_\xi,u_\eta',u_\xi')=G(\eta',\xi')-G(\eta,\xi)$.

4. Analytical continuation of the numerical solution of the integral equation

In the complex plane w arising from the mapping $\alpha = -k\cos w$ the following properties hold [3]: 1) the plus functions in the α plane are even functions in the w plane, 2) the minus function in the α plane are even functions in the plane $w+\pi$.
Taking into account these properties, (6) can be rewritten in the following form:

$$\hat{G}(w_\eta, w_\xi)\hat{F}_{++}(w_\eta, w_\xi) = \hat{F}^i_{++}(w_\eta, w_\xi) + \hat{V}_{++}(w_\eta+\pi, w_\xi+\pi) + \hat{Y}_{++}(w_\eta+\pi, w_\xi) + \hat{Z}_{++}(w_\eta, w_\xi+\pi) \qquad (10)$$

Mathematical manipulations yield the following two-dimensional recursive equations:

$$\hat{G}(w_\eta-\pi, w_\xi)\hat{F}_{++}(w_\eta-\pi, w_\xi) - \hat{G}(-w_\eta-\pi, w_\xi)\hat{F}_{++}(w_\eta+\pi, w_\xi) = \hat{Z}_{++}(w_\eta-\pi, w_\xi+\pi) - \hat{Z}_{++}(w_\eta+\pi, w_\xi+\pi) \qquad (11)$$

$$\begin{aligned}&\hat{G}(w_\eta, w_\xi)\hat{F}_{++}(w_\eta, w_\xi) = \hat{G}(-w_\eta, w_\xi)\hat{F}_{++}(w_\eta+2\pi, w_\xi) + \\ &\hat{G}(w_\eta, -w_\xi-2\pi)\hat{F}_{++}(w_\eta, w_\xi+2\pi) - \hat{G}(-w_\eta-2\pi, -w_\xi-2\pi)\hat{F}_{++}(w_\eta+2\pi, w_\xi+2\pi)]\end{aligned} \qquad (12)$$

If we perform the analytical continuation in w_η, we need to chose the branches of $\hat{G}(w_\eta, w_\xi)$ such that: $\hat{G}(w_\eta, w_\xi) = -k\sin(w_\eta)\sqrt{1-\dfrac{\cos^2 w_\xi}{1-\cos^2 w_\eta}}$. While performing the analytical continuation in w_ξ, the proper braches of $\hat{G}(w_\eta, w_\xi)$ are given by $\hat{G}(w_\eta, w_\xi) = -k\sin(w_\xi)\sqrt{1-\dfrac{\cos^2 w_\eta}{1-\cos^2 w_\xi}}$.
We note that using other possible definitions of angular complex planes w, we can get meromorphic representation of $G(\eta,\xi)$. For instance if $\eta = -k\cos w_1$, $\xi = -\tau_\eta \cos w_2$, $\tau_\eta = -k\sin w_1$ we get: $G(\eta,\xi) = k\sin w_1 \sin w_2$.
Recursive equations hold also in this case.

5. Suggested procedure for solving the problem

First we tray to get a starting spectrum of the plus unknown solving the Fredholm equation (8) in the angular complex planes w_η and w_ξ. We remark that the solution via quadrature is very onerous. We suggest to obtain the starting spectrum by resorting to the representation to integral equation with starting guess the physical optic solution.
Using the recursive equation (12) we can obtain the spectrum of $\hat{F}_{++}(w_\eta, w_\xi)$ in the whole planes w_η and w_ξ.
A final problem concerns with the evaluation of far field via the saddle point evaluation of double integrals [4].

8. References

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