Electromagnetic scattering and radiation of waves by a thin toroidal tube with semi-transparent walls

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Abstract

The work deals with a scalar model of wave scattering and radiation by a thin toroidal tube with the varying longitudinal “conductivity”. The walls of the toroidal tube are assumed to be semi-transparent for the waves. The corresponding boundary conditions represent a direct scalar analog of the electromagnetic boundary conditions describing electromagnetic properties of carbon nanotubes (CNTs) recently discussed in the literature. The method of matching of asymptotic expansions is applied to study the model problem provided that the length of the tube is much greater than its thickness. The leading term of the formal asymptotics is developed and analyzed. The scattering amplitude is then considered and its resonant properties are discussed.

1. Formulation. Physical reasoning

In the present work we study scattering and radiation of waves by a thin hollow tube of the finite length which is rolled up into a torus. The torus has a convex cross-section of the tube ω_ε with the diameter 2ε which is much less than the bending radius $a = 1$, $0 < \varepsilon \ll 1$.

The surface of such a toroidal tube is semi-transparent for the waves, which means that the boundary conditions on its surface are the two-side boundary conditions of impedance type (see below the statement of the model problem). Moreover, the surface impedance along the tube is assumed to be much grater than that in the transverse direction. The longitudinal impedance varies along the tube and is discontinuous at the points of junction. Actually, the wave processes are governed by the scalar Helmholtz equation and the corresponding boundary value problem is considered as a simplified model for the whole vector electromagnetic problem of scattering by a carbon nano-tube (CNT).

The problem of electromagnetic scattering and radiation by CNTs has recently attracted attention of researches because of their potential use as nano-antennas and electronic waveguides, see e.g. [1],[2]. The electromagnetic properties of the CNTs are described by means of the macroscopic electromagnetic boundary conditions on the surface of a CNT. The latter conditions have the form of impedance-sheet boundary conditions (see also [6], chap. 1)

$$H_\phi|_{\rho=a+0} - H_\phi|_{\rho=a-0} = \frac{4\pi\sigma_{zz}}{c}\, E_z|_{\rho=a}$$

and the other tangential field components are continuous across the cylindrical surface $\rho = a$, $0 \le \phi < 2\pi$, $|z| < \infty$ (see [1]), where σ_{zz} is the large longitudinal conductivity of a metallic CNT. The conductivity σ_{zz} is complex valued and is derived from the microscopic quantum considerations [1]. Remark that the toroidal CNTs are also of interest, at least, from the theoretical point of view [3]. The electromagnetic wave scattered or radiated by a CNT also satisfies Maxwell’s equations, radiation conditions at infinity as well as the Meixner’s type conditions at the edges of the structure.[1]

2. Statement of the model problem

Consider a 2-D bounded simple connected convex domain ω and the corresponding small domain $\omega_\varepsilon = \{y = (y_1, y_2) \in R^2 :\ (\varepsilon^{-1}y_1, \varepsilon^{-1}y_2) \in \omega\}$. In particular, we can take $\omega_\varepsilon = \{(y_1, y_2) \in R^2 : y_1^2 + y_2^2 < \varepsilon^2\}$. Let the contour M be a circumference in $\mathbf{R}^3$, $M = \{(x_1, x_2, x_3) :\ x_3 = 0,\ x_1^2 + x_2^2 = 1\}$, $x = (x_1, x_2, x_3)$ and $\varphi \in [0, 2\pi)$ be a natural angular parameter along M. In a standard manner, we attribute the cylindrical system of coordinates (z, ρ, φ) to the Cartesian coordinates (x_1, x_2, x_3) and specify the connection $y_1 = z,\ y_2 = \rho - 1,\ \zeta = \varphi$. Remark that $\rho = 1$ on M.

[1]The electromagnetic problem in its complete formulation will be considered elsewhere.

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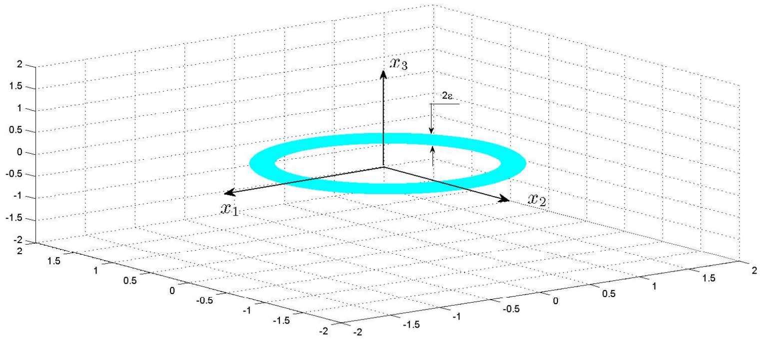

Fig.1 Wave scattering by a thin tube

Define a thin toroidal domain $\Omega^i_\varepsilon = \{(\varphi, y_1, y_2) \in R^3 : \ \varphi \in M, \ y \in \omega_\varepsilon\}$ with the boundary S_ε, Fig. 1. Let $\Omega^e_\varepsilon \subset \mathbf{R}^3$ be the exterior of the surface S_ε. The wave field $U(x, \varepsilon)$ fulfills the Helmholtz equation

$$(\triangle + k^2)U(x, \varepsilon) = f(x), \quad x \in \Omega^e_\varepsilon \tag{1}$$

and

$$(\triangle + k^2)U(x, \varepsilon) = 0, \quad x \in \Omega^i_\varepsilon, \tag{2}$$

where f is a source function with a compact support in Ω^e_ε, $k > 0$. (The time dependence $\exp(-\mathrm{i}\omega_0 t)$ is assumed and suppressed throughout the paper.)

The boundary conditions on the surface S_ε are satisfied

$$[U](x, \varepsilon) = 0, \quad x \in S_\varepsilon \tag{3}$$

and

$$\left[\frac{\partial U}{\partial n}\right](x, \varepsilon) = \frac{\alpha(\varphi)}{\varepsilon}\left(U(x, \varepsilon) + U_0(x, \varepsilon)\right), \quad x \in S_\varepsilon, \tag{4}$$

where

$$[H](x, \varepsilon) = H|_{x \to S_\varepsilon,\, x \in \Omega^e_\varepsilon} - H|_{x \to S_\varepsilon,\, x \in \Omega^i_\varepsilon}$$

is the difference of the limiting values of H on S_ε from the exterior and from the interior correspondingly. The known function U_0, an asymptotic series with respect to ε, specifies a given surface current function on S_ε[2] and n is the external to Ω^i_ε normal vector.

The coefficient $\frac{\alpha(\varphi)}{\mathrm{i}\varepsilon}$ is the surface impedance depending only on the angular coordinate φ. Obviously, the surface impedance is assumed to be large and explicitly dependent on the parameter ε, $\alpha(\varphi)$ is continuous as $\varphi \in [0, 2\pi]$ and $\alpha(0 + 0) \neq \alpha(2\pi - 0)$. In other words, the surface impedance has the discontinuity of the first kind on S_ε so that the curve l_ε of junction on S_ε corresponding to $\varphi = 0$ (or $\varphi = 2\pi$) is the line of the impedance jump. The classical formulation of the problem implies also formulation of a Meixner's type condition in a vicinity of the the curve l_ε.

The radiation condition implies that the wave field has the following asymptotics at infinity

$$U(x, \varepsilon) - v(x) = (-2\pi)\, \mathcal{A}(\theta, \varphi, \varepsilon)\, \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}\left(1 + O(1/kr)\right), \quad r \to \infty, \tag{5}$$

where

$$v(x) = -\frac{1}{4\pi} \int\limits_{R^3} \frac{\mathrm{e}^{\mathrm{i}k\|x-\zeta\|}}{\|x - \zeta\|}\, f(\zeta)\, d\zeta,$$

is the incident wave generated by the source $f(x)$, $\mathcal{A}(\theta, \varphi, \varepsilon)$ is the scattering amplitude, $(x_1, x_2, x_3) = (r\cos\varphi\, \sin\theta, r\sin\varphi\, \sin\theta, r\cos\theta)$. The leading term $A(\theta, \varphi)$ (with respect to ε) of the scattering amplitude is one of the most important characteristics to be computed herein. We consider classical solution of the problem (1)–(5) and look for the leading term of its asymptotics as $\varepsilon \to 0+$.

[2]Provided $f = 0, \ U_0 \neq 0$, it is natural to speak about the radiation of waves generated by the surface current on the tube and, vice a versa, $f \neq 0, \ U_0 = 0$ correspond to the scattering of waves by the tube.

3. Formal asymptotics

In construction of an asymptotic solution one of the most important step is in the formulation of the asymptotic ansatz which is an asymptotic expression for the solution of a problem at hand. It has a specific form describing the characteristic behavior of the solution as the small parameter ε tends to its limiting value. The ansatz may depend on some stretched or reduced variables and contain unknown functions which are usually determined by solving a set of the so called limiting problems (see, e.g. [4]). In our case we can exploit the experience gained from similar problems (e.g. [5], [4], chapter 12) and also from some physical intuition dealing with the physical reasoning of the solution behavior.

The wave field $v(x)$, which is considered as an incident wave, interacts with the scatterer M that is the limiting set representing S_ε. As a result of simple physical arguments, the scattered by M field is given by the potential

$$V(x;\gamma) = -\frac{1}{4\pi}\int\limits_M \frac{\mathrm{e}^{\mathrm{i}k\|x-(\tau,0)\|}}{\|x-(\tau,0)\|}\,\gamma(\tau)\,dl_\tau,$$

with the unknown yet density $\gamma(\tau)$ defined on M,[3] l_φ is the arc-length with the parameter φ along M and $(\varphi,0) := (\varphi,y_1,y_2)|_{y=0}$ by definition.

Although the sum $v(x)+V(x;\gamma)$ satisfies the Helmholtz equation (1) (as well as the radiation condition) in the exterior domain, it gives a non-trivial discrepancy in the boundary conditions. The latter should be compensated by the boundary-layer functions. In particular, the boundary value problems for boundary layers are written and studied as well as the integral equation for the current $\gamma(\varphi)$.

4. The particular case of the homogeneous toroidal tube

In the present section we shall assume that the impedance of the tube $\frac{\alpha(\varphi)}{\mathrm{i}\varepsilon}$ is continuous $2\pi-$periodic function of φ or, even more simple, $\alpha(\varphi) = const$ and the tube is right-circular. The latter assumptions enable one to construct the boundary layers explicitly and to solve the integral equation for γ. In this case the leading terms of the asymptotics have form

$$U(x,\varepsilon) \sim v(x) + V(x;\gamma) + \chi(|y|)w(\varphi, y/\varepsilon). \tag{6}$$

In the interior domain Ω^i_ε the leading term of the asymptotics is

$$U(x,\varepsilon) \sim w(\varphi, y/\varepsilon)\,. \tag{7}$$

The 3-D boundary layer disappears from the construction. The problem for the boundary layer w in (6), (7) can be constructed by separation of variables. In what follows, we write down the formula for the 'current' $\gamma(\varphi)$. Finally, a relatively simple formula for the scattering amplitude will be obtained in this case.

4.1 Integral equation and the scattering amplitude

In accordance with (5),(6) the far field is specified by $v(x)+V(x;\gamma)$ and the leading term of the scattering amplitude takes the form

$$\mathcal{A}(\theta,\varphi,\varepsilon) \sim A(\theta,\varphi) = \frac{k}{8\pi^2}\int\limits_0^{2\pi} \gamma(\tau)\,\mathrm{e}^{-\mathrm{i}k\sin\theta\cos[\tau-\varphi]},\quad \varepsilon\to 0\,,$$

where (θ,φ) are the angles of the spherical system of coordinates attributed to the Cartesian coordinates $x=(x_1,x_2,x_3)$.

The integral equation for γ now reads

$$\left(\alpha^{-1} - \log\left(\frac{\varepsilon}{8}\right)\right)\gamma(\varphi) + \frac{1}{2}\int\limits_M \frac{\gamma(\tau)-\gamma(\varphi)}{2|\sin([\tau-\varphi]/2)|}\,d\tau + \frac{1}{2}\int\limits_M \frac{\mathrm{e}^{2\mathrm{i}k|\sin([\tau-\varphi]/2)|}-1}{2|\sin([\tau-\varphi]/2)|}\,\gamma(\tau)\,d\tau \;=\; g(\varphi) \tag{8}$$

[3]The function γ plays the role of the induced current on M.

and is solved in a simple form. We represent $g(\varphi)$ and $\gamma(\varphi)$ by their Fourier series $\gamma(\varphi) = \sum_{p=-\infty}^{+\infty} \Gamma_p \, e^{ip\varphi}$, $g(\varphi) = \sum_{p=-\infty}^{+\infty} g_p \, e^{ip\varphi}$, substitute them into the equation (8) and, after simple calculations, obtain

$$\gamma(\varphi) = -\sum_{p=-\infty}^{+\infty} \frac{g_p \, e^{ip\varphi}}{[-\alpha^{-1} + \log\left(\frac{\varepsilon}{8}\right) + \lambda_{|p|} + \mu_p(k)]} \, ,$$

where $\lambda_p = 2\sum_{j=1}^{|p|-1}(2j+1)^{-1}$, $-2\mu_m(k) = \int_0^{2\pi} \frac{e^{2ik|\sin(\tau/2)|}-1}{2|\sin(\tau/2)|} e^{ip\tau} \, d\tau$ are the eigenvalues of the integral operators correspondingly. The latter formula is substituted into the expression for the scattering diagram $A(\theta, \varphi)$. Changing the orders of integration and summation, which is justified, we arrive at the expression

$$A(\theta, \varphi) = -\frac{k}{4\pi} \sum_{p=-\infty}^{+\infty} \frac{g_p \, e^{ip[\varphi+\pi/2]} J_p(-k \sin\theta)}{[-\alpha^{-1} + \log\left(\frac{\varepsilon}{8}\right) + \lambda_{|p|} + \mu_p(k)]} \, , \tag{9}$$

where the integral representations $J_n(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(z\sin t - nt)} \, dt$ for the Bessel function was exploited. The denominator $D_p(k, \log\varepsilon) = -\alpha^{-1} + \log\left(\frac{\varepsilon}{8}\right) + \lambda_{|p|} + \mu_p(k)$ in (9) may have complex zeros as a function of k which can be naturally called resonances. In the problems of scattering the most interesting are those located near the real axis $\mathrm{Im}\, k = 0$.

4.2 Resonances and numerical simulation

In the report we shall demonstrate
1. Numerical determination of the resonances, position of the singularities of $[D_p(k, \log\varepsilon)]^{-1}$ on the complex plane $k \in C$, say, for two different values of ε
2. The results of numerical calculation of $|A(\theta, \varphi)|$ as a function of θ and φ for the non-resonant and resonant values of k
3. Spherical diagram of the modulus of the scattering amplitude $|A(\theta, \varphi)|$

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References

[1] Maksimenko S A and Slepyan G Ya 2000 Electrodynamic Properties of Carbon Nanotubes in *Electromagnetic Fields in Unconventional Materials and Structures* Edited by Singh and Lakhtakia, ISBN 0-471-33656-1, John Wiley & Son Inc

[2] Slepyan G Ya, Shuba M V, Maksimenko S A and Lakhtakia A 2006 Theory of optical scattering by achiral carbon nanotubes and their potential as optical nanoantennas *Physical Review B* 73(19) 19516

[3] Lei Liu, Guo G Y, Jayanthi C S and Wu S Y 2002 Colossal Paramagnetic Moments in Metallic Carbon Nanotori *Phys Rev Lett* 88 217206

[4] Maz'ya V G, Nazarov S A and Plamenevsij B A 2000 *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains* Vol I, II Birkhauser-Verlag Basel

[5] Maz'ya V G, Nazarov S A and Plamenevsij B A 1981 Asymptotics of solutions of the Dirichlet problem in the a domain with a cut-out thin tube *Matematicheskij sbornik* 2 (10) pp 187-217

[6] Buldyrev V S and Lyalinov M A 2001 *Mathematical methods in modern electromagnetic diffraction theory*, Vol 1, Intern. monographs on advanced electromagnetics, Science House Tokyo p. 242