Location and density determination for a source with impulse derivative time variation

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Abstract

In this study an inverse source problem related to the source density $f_0(\mathbf{x})$ which is a function of bounded support and taking place in the wave equation $\Delta u(\mathbf{x}, t) - (1/c^2)\, \partial^2 u(\mathbf{x}, t) / \partial t^2 = - f_0(\mathbf{x})\delta'(t)$ is considered. An explicit expression of the solution is given in terms of the surface integral of the data which is measured on the boundary S of a *convex* domain D during certain *finite* time interval [0,T]. An illustrative example shows the applicability as well as the effectiveness of the method.

1. Introduction

The aim of this work is to determine the source density $f_0(\mathbf{x})$ which appears in the wave equation given by

$$\Delta u(\mathbf{x}, t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} u(\mathbf{x}, t) = - f_0(\mathbf{x})\, \delta'(t). \tag{1}$$

Here $\mathbf{x} = (x_1, x_2, x_3) \in R^3$ and $t \in R$ stand for any point in the three-dimensional space and the time, respectively, while c is a positive constant which signifies the propagation velocity of the wave. In the problem to be considered here, $f_0(\mathbf{x}) \in L^1(V_0)$ is assumed to be a function of bounded support with support $V_0 \subset R^3$ and $\delta'(t)$ is the derivative of the classical delta-Dirac distribution $\delta(t)$ with respect to time t. As it is well known in a direct problem the source density function $f_0(\mathbf{x})$ is assumed to be known and one tries to find the wave function $u(\mathbf{x},t)$. But in the inverse source problem which is considered here, the situation is converse: one assumes that the wave function $u(\mathbf{x},t)$ is known (by measurements) during certain time interval [0,T] on the boundary S of a convex domain D which involves the region V_0 inside, and tries to determine the source density $f_0(\mathbf{x})$. The result will also reveal the support V_0 where the source is located. An explicit expression of the solution is given in terms of the surface integral of the data on S which can be performed numerically by the method of discretization. It is also worthwhile to notice that the measured data on S are collected synthetically by solving the corresponding direct problem. In order to show the applicability as well as the effectiveness of the method an illustrative example is considered. The inverse problem stated above is motivated by the so-called photo-acoustic and thermo-acoustic tomography which has several important applications in medicine [1-3].

Since the solution of the inverse problem is always based on the solution to the direct problem, one will first consider the direct problem and derive an explicit expression for its solution in the following section 2. Then in section 3 the exact results pertinent to the inverse problem will be obtained. In section 4 an appropriate algorithm to numerical application is proposed and then an illustrative example which shows the applicability as well as the effectiveness of the method is given in section 5. Finally, in section 6 one summarizes the main results obtained in this work and some open problems to the forthcoming investigations are discussed.

2. Solution of the direct problem

By taking the Fourier transform of both sides of (1) with respect to t one gets

$$\Delta \hat{u}(\mathbf{x},\omega) + k^2 \hat{u}(\mathbf{x},\omega) = i\omega f_0(\mathbf{x}), \qquad \mathbf{x} \in R^3, \tag{2a}$$

where $\hat{u}(\mathbf{x},\omega)$ and k are defined respectively by

$$\hat{u}(\mathbf{x},\omega) = \int_{-\infty}^{\infty} u(x,t)\, e^{i\omega t} dt, \quad \omega \in (-\infty, \infty) \tag{2b}$$

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and

$$k = \omega / c. \tag{2c}$$

To write the right hand side of (2a) one considers that the Fourier transform of $\delta'(t)$ is given by

$$\hat{\delta}' = - i\omega. \tag{2d}$$

Besides the basic equation (2a) one has to consider also the radiation conditions written by

$$\hat{u}(\mathbf{x},\omega) = O(1 / |\mathbf{x}|) \text{ as } |\mathbf{x}| \to \infty \tag{3a}$$

and

$$\frac{\partial \hat{u}}{\partial |\mathbf{x}|}(\mathbf{x},\omega) - i k\, \hat{u}(\mathbf{x},\omega) = o(1 / |\mathbf{x}|) \text{ as } |\mathbf{x}| \to \infty. \tag{3b}$$

By using the well known method of Green function, the solution related to equation (2a) under the conditions (3a-b) can easily be obtained as below:

$$\hat{u}(\boldsymbol{\eta},\omega) = - i\,\omega \int_{V_0} f_0(\mathbf{x})\, G_1(\mathbf{x}, \boldsymbol{\eta}) d\mathbf{x}, \qquad \boldsymbol{\eta} \in R^3. \tag{4}$$

Here $\boldsymbol{\eta}$ stands for any point in the space and $G_1(\mathbf{x}, \boldsymbol{\eta})$ is the Green function defined by

$$G_1(\mathbf{x}, \boldsymbol{\eta}) = \frac{1}{4\pi} \frac{e^{ikR}}{R}, \qquad R = |\mathbf{x} - \boldsymbol{\eta}|. \tag{5a}$$

It is an easy matter to check that $G_1(\mathbf{x}, \boldsymbol{\eta})$ satisfies the Helmholtz equation

$$\Delta G + k^2 G = - \delta(\mathbf{x} - \boldsymbol{\eta}) \tag{5b}$$

and the following radiation conditions :

$$G_1 = O(1/R) \quad , \quad \frac{\partial G_1}{\partial R} - i k G_1 = O(1/R^2) \quad \text{as} \quad R \to \infty. \tag{5c,d}$$

It is obvious that (4) is the solution to the direct problem in frequency domain which will be used to establish a formula for the solution to the inverse problem.

3. Solution of the inverse problem

Before going further into the details of the inverse problem solution, a lemma which we will be used later on will be given without proof. To this end consider a convex domain $D \subset R^3$ and two arbitrarily chosen points $\mathbf{x} \in D$ and $\mathbf{y} \in D$. Then define a polar coordinate system (r, θ, ϕ) with the origin at the mid-point $(\mathbf{x} + \mathbf{y}) / 2$ and polar axis parallel to the

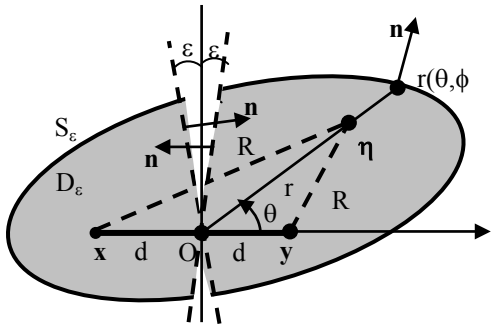

Fig.1. A convex domain D_ε and its boundary S_ε

vector $(\mathbf{y} - \mathbf{x})$ (see Fig.-1). Let a two-part region $D_\varepsilon \subset D$ be defined as follows:

$$D_\varepsilon = D \cap \{(r,\theta, \phi) : r \in [0,\infty),\ \theta \in [0, \pi/2 - \varepsilon] \cup [\pi/2 + \varepsilon, \pi],\ \phi \in [0, 2\pi)\}. \tag{6}$$

Here $\varepsilon > 0$ stands for an arbitrary small angle. Then one can state the following lemma:

Lemma: Let $R_1 = |\mathbf{y} - \boldsymbol{\eta}|$ and $R_2 = |\mathbf{x} - \boldsymbol{\eta}|$ denote the distances of any point $\boldsymbol{\eta} \in D_\varepsilon$ to the points $\mathbf{y}$ and $\mathbf{x}$, respectively. Then one has

$$\int_{D_\varepsilon} \frac{\delta'\{(R_2 - R_1)/c\}}{R_1 R_2} dV_\eta = 0. \tag{7}$$

Here dV_η stands for the volume element at the point $\boldsymbol{\eta}$.

Now in order to find an explicit expression for the density function $f_0(x)$ of the source appearing in (1) the following theorem should be considered.

Theorem: Let the values of the field $u(\mathbf{x},t)$ be known on the boundary S of a convex domain D for all $t\in[0,T]$ where $T = R_{max}/c \equiv \max |\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2|/c$ with $\boldsymbol{\eta}_{1,2} \in S$. Then the density of the source can be determined uniquely trough the formula given by

$$f_0(\mathbf{x}) = \frac{1}{2\pi c^2} \int_S \left[\frac{1}{R^2} u(\boldsymbol{\eta}, \frac{R}{c}) - \frac{1}{cR} \frac{\partial}{\partial t} u(\boldsymbol{\eta}, \frac{R}{c})\right] \cos\varphi \, dS_\eta. \tag{8}$$

Here $R = |\mathbf{x} - \boldsymbol{\eta}|$ denotes the distance between the points $\mathbf{x}$ and $\boldsymbol{\eta}$ while φ is the angle between the outward normal to S at the point $\boldsymbol{\eta}$ and the vector $(\boldsymbol{\eta}-\mathbf{x})$ (see Fig.-2).

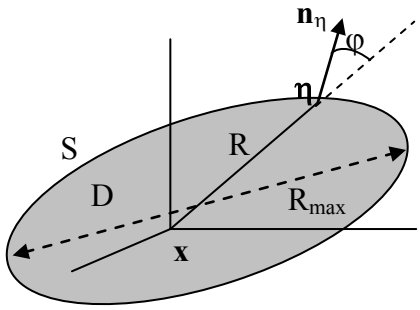

Fig.2. The boundary S of a convex domain D which contains the source

For the sake of simplicity of the paper the proof of this theorem is omitted here. But at least it is worthwhile to notice that by taking into account the Green function $G_2(\mathbf{x},\boldsymbol{\eta})$ defined by

$$G_2(\mathbf{x}, \boldsymbol{\eta}) = \frac{1}{4\pi} \frac{e^{-ikR}}{R} \quad , \qquad R = |\mathbf{x} - \boldsymbol{\eta}| \tag{9a}$$

which meets the equation (5b) with the radiation conditions given by

$$G_2 = O(1/R) \quad , \quad \frac{\partial G_2}{\partial R} + i\,k\,G_2 = O(1/R^2) \quad \text{as} \quad R \to \infty \tag{9b}$$

and using also the well known Gauss-Ostrogradski theorem with the consideration of the above cited lemma it is not too complicated to prove this theorem.

4. A numerical algorithm for practical applications

In practical applications the integration in formula (8) can only be performed by discretization. To this end one chooses certain reasonably distributed discrete points $\mathbf{x}_\alpha$ ($\alpha = 1, \dots, N_\alpha$) inside the region D and $\boldsymbol{\eta}_\beta$ ($\beta = 1, \dots, N_\beta$) on the surface S. Thus R/c becomes equal to certain quantities $t_{\alpha\beta} = |\boldsymbol{\eta}_\beta - \mathbf{x}_\alpha| / c = R_{\alpha\beta}./c$. A small number τ_α can be fined such that $\tau_\alpha << t_{\alpha\beta}$ and

$$\frac{\partial}{\partial t} u(\boldsymbol{\eta}, t_{\alpha\beta}) \approx \frac{u(\boldsymbol{\eta}, t_{\alpha\beta} + \tau_\alpha) - u(\boldsymbol{\eta}, t_{\alpha\beta} - \tau_\alpha)}{2\tau_\alpha}$$

provides an acceptable approximation for the time derivatives. Thus (8) becomes reduced to the sum

$$f_0(\mathbf{x}_\alpha) = \frac{1}{2\pi c^2} \sum_{\beta=1}^{N_\beta} \left\{ \frac{1}{R_{\alpha\beta}^2} u(\boldsymbol{\eta}_\beta, t_{\alpha\beta}) - \frac{1}{2c\tau_\alpha R_{\alpha\beta}} \left[u(\boldsymbol{\eta}_\beta, t_{\alpha\beta} + \tau_\alpha) - u(\boldsymbol{\eta}_\beta, t_{\alpha\beta} - \tau_\alpha) \right] \right\} \cos\varphi_{\alpha\beta} \, \Delta S_\beta, \tag{10}$$

where $\alpha = 1, \dots, N_\alpha$. Here ΔS_β stands for (approximate) area element around the point $\boldsymbol{\eta}_\beta$ while $\cos\varphi_{\alpha\beta}$ is the cosine of the angle between the normal vector to S at $\boldsymbol{\eta}_\beta$ and the vector $(\boldsymbol{\eta}_\beta - \mathbf{x}_\alpha)$.

5. An illustrative example

In order to see the applicability as well as the accuracy of the theory established above one considers a simple illustrative example in which the source is assumed to be constant and homogeneously distributed in a

sphere with radius a. Then by considering that the volume density value of the source is equal to 1 the source density function $p_0(\mathbf{x})$ is expressed as $f_0(\mathbf{x}) = H(a - r)$ where H(.) being Heaviside unit step function and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The surface S where the measurements will be performed is assumed to be a sphere which is concentric with the spherical source distribution and has a radius $\eta > a$. The data which should be collected by measurements on S are obtained synthetically by solving the corresponding direct problem. Since the computations are straightforward, we write directly the result:

$$u(\eta,t) = \frac{c^2}{2}\left(1-\frac{ct}{\eta}\right)\left[H\left(t-\frac{\eta-a}{c}\right)-H\left(t-\frac{\eta+a}{c}\right)\right]. \quad (11)$$

The results which are obtained by the consideration of (11) in the discrete expression given by (10) for 8 different observation points are presented in Table 1. In these applications the radii of the spherical source and measurement surface S are assumed as $a = 1$ and $\eta = 3$, respectively. In the table given below the first column shows the coordinates $x_{j\alpha}$ (j = 1,2,3) and $r_\alpha = \sqrt{x_{1\alpha}^2 + x_{2\alpha}^2 + x_{3\alpha}^2}$ for the observation points $\mathbf{x}_\alpha$ (α= 1, …, 12). The second column presents small difference parameter $c\tau_\alpha$ related to $\mathbf{x}_\alpha$. Finally, the other four columns show the computed values of source density function $f_0(\mathbf{x}_\alpha)$ for different discretization numbers N_β.

Table 1. Computed values at randomly chosen points inside and outside the source region

$\mathbf{x}_\alpha=(x_{1\alpha}, x_{2\alpha}, x_{3\alpha})$ r_α	$c\tau_\alpha$	$f_0(\mathbf{x}_\alpha)$ for $N_\beta=10^2$	$f_0(\mathbf{x}_\alpha)$ for $N_\beta=10^4$	$f_0(\mathbf{x}_\alpha)$ for $N_\beta=10^6$	$f_0(\mathbf{x}_\alpha)$ for $N_\beta=10^8$
$\mathbf{x}_1$=(0 , 0 , 0) r_1= 0	3 x 10^{-3}	0.9876884	0.9998766	0.9999998	1.0000002
$\mathbf{x}_2$=(-0.1 , -0.2 , 0.3) r_2= 0.3742	2.626 x 10^{-3}	0.9875105	0.9998748	0.9999992	1.0000001
$\mathbf{x}_3$=(0.4 , 0.5 , - 0.2) r_3= 0.6708	2.33 x 10^{-3}	0.9881372	0.9998811	0.9999987	0.9999998
$\mathbf{x}_4$=(0.7 , 0.5 , 0.4) r_4= 0.9487	2.0513 x 10^{-3}	0.9883264	0.9998829	0.9999987	0.9999999
$\mathbf{x}_5$=(-0.9 , 0.8 , 0.6) r_6= 1.3453	1.655 x 10^{-3}	0.6901003	-0.1365812	0.0022651	-0.0000879
$\mathbf{x}_6$=(-1.3 , -1.1 , 0.9) r_8= 1.9261	1.074 x 10^{-3}	0.4224414	0.1108369	0.0034242	0.0000516
$\mathbf{x}_7$=(-1.4 , 1.6, -1.3) r_{10}= 2.4919	0.508 x 10^{-3}	0.2559248	0.0845118	0.0138711	0.0000648
$\mathbf{x}_8$=(1.6 , -1.5, 1.9) r_{12}= 2.9017	0.0983 x 10^{-3}	0.1896277	-0.0531573	0.0110534	0.0003573

6. Conclusions and concluding remarks

From the results obtained above one concludes that by using the data obtained by measurements performed on a surface S (which limits a convex volume D) during a finite time interval one can determine the density of a source of bounded support inside D. Since the result determines also the support of the source, it can be used in the detection of regions in biological mediums, which are sensible to microwave electromagnetic energy. In the present work one has considered the case where there is no (or negligibly small) reflections on the boundary S. If the reflections on S are not negligible, then the analysis must be performed by considering transmission or impedance type boundary conditions on S.

7. References

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