Estimation of the Parameters of 2D Debye Dispersive Media Using a Time-Domain Inverse Scattering Technique

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Abstract

A time-domain inverse scattering method for the reconstruction of inhomogeneous dispersive media described by the Debye model is presented. The estimation of the parameters characterizing the scatterer is based on the minimization of a cost function, which describes the discrepancy between measured and estimated values of the electric field. Applying the calculus of variations, we derive the Fréchet derivatives with respect to the scatterer properties, which can be utilized by any gradient-based optimization technique. Numerical results for the reconstruction of two-dimensional Debye scatterer using the Polak-Ribière algorithm exhibit the efficiency of the proposed method.

1. Introduction

There is considerable scientific interest, nowadays, in the study of the electromagnetic inverse scattering problem of microwave imaging, due to its applications in medical imaging, nondestructive testing, geophysical prospecting etc. Consequently, various time-domain inverse scattering methods [1-3] have been developed in order to cope with this problem. The goal of the methods is to estimate the unknown scatterer properties by taking into account and inverting time-domain field measurements when the scatterer is illuminated by wideband incident waves. In the case of dispersive media, the scatterer properties are frequency-dependent, hence dispersion phenomena appear and the problem becomes a bit more complex. Obviously, the development of methods for the estimation of the properties of dispersive media is crucial [4, 5].

In the present work, a time-domain inverse method for the simultaneous reconstruction of the relative optical and static permittivities as well as the relaxation time of Debye scatterers is proposed. According to the method, a cost function that describes the difference between measured and estimated values of the electric field, is minimized. The Fréchet derivatives of the cost function with respect to the scatterer properties are derived analytically. Then, these derivatives are utilized by the Polak-Ribière nonlinear optimization algorithm to estimate, through an iterative procedure, the spatial distribution of the scatterer properties. Note that the FDTD method is employed for the solution of the direct and the adjoint electromagnetic scattering problem.

2. Mathematical Formulation of the Problem

The relative complex permittivity of an inhomogeneous scatterer exhibiting Debye dispersion is given by

$$\varepsilon_r(\omega,\mathbf{r}) = \varepsilon_\infty(\mathbf{r}) + \frac{\Delta\varepsilon(\mathbf{r})}{1 + j\omega\tau(\mathbf{r})} \tag{1}$$

where ε_∞ is the optical relative permittivity, τ is the relaxation time, $\Delta\varepsilon = \varepsilon_s - \varepsilon_\infty$ (ε_s is the static relative permittivity) and ω is the angular frequency. We assume that the Debye scatterer is non-magnetic ($\mu = \mu_0$) and occupies the scatterer domain D. The domain is illuminated by I incident wideband waves and for each one of them, the electric field is measured at K positions around the scatterer for the time interval $[0, T]$. Thus, a set of $I \times K$ electric field measurements is obtained, which are denoted as $\mathbf{E}_{ik}^m$ where $i = 1,...,I$ and $k = 1,...,K$.

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The original unknown properties of the medium are estimated by inverting the measurements $\mathbf{E}_{ik}^{m}$. This can be succeeded through the minimization of the following cost function

$$\begin{aligned} F(\mathbf{p},\mathbf{E},\mathbf{H},\mathbf{J},\mathbf{e},\mathbf{h},\mathbf{q}) &= \frac{1}{2}\sum_{i=1}^{I}\sum_{k=1}^{K}\int_0^T \left\|\mathbf{E}_{ik}-\mathbf{E}_{ik}^{m}\right\|^2 dt \\ &+\sum_{i=1}^{I}\int_0^T\int_V \begin{bmatrix} \mathbf{h}_i\cdot\left(\nabla\times\mathbf{E}_i+\mu\partial_t\mathbf{H}_i\right)+\mathbf{e}_i\cdot\left(\nabla\times\mathbf{H}_i-\varepsilon_0\varepsilon_\infty\partial_t\mathbf{E}_i-\mathbf{J}_i-\mathbf{J}_{si}\right) \\ +\mathbf{q}_i\cdot\left(\mathbf{J}_i+\tau\partial_t\mathbf{J}_i-\varepsilon_0\Delta\varepsilon\partial_t\mathbf{E}_i\right) \end{bmatrix} dvdt \end{aligned} \tag{2}$$

where $\mathbf{p}=[\varepsilon_\infty,\Delta\varepsilon,\tau]^T$ is an estimate of the medium properties, $\mathbf{E}_{ik}$ is the estimated electric field (for a given estimate of $\mathbf{p}$) at the *k*th measurement position for the *i*th incidence, $\mathbf{J}_{si}$ is the current density that generates the *i*th incidence and $\mathbf{J}_i$ is the current density that arises inside the scatterer. Moreover, V is the domain of computation and T is the time duration of measurement for each incidence. As it is clearly depicted in (2), the Maxwell's curl equations and the polarization relation are involved as equality constraints through the use of the Lagrange vector multipliers $\mathbf{e}_i$, $\mathbf{h}_i$ and $\mathbf{q}_i$.

The calculus of variations dictates that the necessary condition for (2) to be minimized is that its first-order variation δF, must be equal to zero. As a consequence, the Lagrange vector multipliers satisfy the following relations:

$$\nabla\times\mathbf{e}_i-\mu\partial_t\mathbf{h}_i=0 \tag{3}$$

$$\nabla\times\mathbf{h}_i+\varepsilon_0\varepsilon_\infty\partial_t\mathbf{e}_i-\mathbf{j}_i+\sum_{k=1}^{K}(\mathbf{E}_{ik}-\mathbf{E}_{ik}^{m})=0 \tag{4}$$

$$\mathbf{j}_i-\tau\partial_t\mathbf{j}_i+\varepsilon_0\Delta\varepsilon\partial_t\mathbf{e}_i=0 \tag{5}$$

$$\mathbf{e}_i\big|_{t=T}=\mathbf{h}_i\big|_{t=T}=\mathbf{j}_i\big|_{t=T}=0 \tag{6}$$

where $\mathbf{j}_i=-\varepsilon_0\varepsilon_\infty\partial_t\mathbf{q}_i$. From the above relations, it is clear that $\mathbf{e}_i$ and $\mathbf{h}_i$ can be considered as waves propagating backwards in time and inwards in space because they satisfy the Maxwell's curl equations and the polarization relation in reversed time.

3. Fréchet Derivatives

From the first-order variation of (2), the Fréchet derivatives with respect to the scatterer properties are derived. For the case of a Debye scatterer, the derivatives are given by

$$\mathbf{G}_{\varepsilon_\infty}=\frac{\delta F}{\delta\varepsilon_\infty}=-\sum_{i=1}^{I}\int_0^T(\mathbf{e}_i\cdot\partial_t\mathbf{E}_i)dt \tag{7}$$

$$\mathbf{G}_{\Delta\varepsilon}=\frac{\delta F}{\delta\Delta\varepsilon}=-\frac{1}{\Delta\varepsilon}\sum_{i=1}^{I}\int_0^T(\mathbf{E}_i\cdot\mathbf{j}_i)dt \tag{8}$$

$$\mathbf{G}_{\tau}=\frac{\delta F}{\delta\tau}=\frac{1}{\varepsilon_0\Delta\varepsilon}\sum_{i=1}^{I}\int_0^T(\mathbf{J}_i\cdot\mathbf{j}_i)dt\,. \tag{9}$$

These derivatives can be utilized by any gradient-based optimization technique for the reconstruction of the unknown parameters. We should note that in the present work, the Polak-Ribière conjugate gradient algorithm has been adopted. If the scatterer domain is partitioned into N subdomains, within which the scatterer properties are

constant, the properties vector $\mathbf{p} = [\varepsilon_\infty^1, ..., \varepsilon_\infty^N, \Delta\varepsilon^1, ..., \Delta\varepsilon^N, \tau^1, ..., \tau^N]^T$ is defined that describes the scatterer. According to the Polak-Ribière inversion algorithm, the vector $\mathbf{p}$ is updated through the following scheme:

$$\mathbf{p}^{(l+1)} = \mathbf{p}^{(l)} + \gamma^{(l)} \mathbf{v}^{(l)} \tag{10}$$

where

$$\mathbf{v}^{(l)} = -\mathbf{G}^{(l)} + \frac{\mathbf{G}^{(l)} \cdot \left(\mathbf{G}^{(l)} - \mathbf{G}^{(l-1)}\right)}{\left\|\mathbf{G}^{(l-1)}\right\|^2} \mathbf{v}^{(l-1)} . \tag{11}$$

Note that the value of γ in (10) is derived by minimizing the cost function along the direction of vector $\mathbf{v}^{(l)}$ (line search optimization) and l denotes the iteration.

4. Numerical Results

The presented inverse scattering technique has been applied to reconstruct a two-dimensional Debye medium. The geometrical configuration of the problem is shown in Fig. 1 where the incident waves are generated by 8 transmitters uniformly distributed around the scatterer, whereas the electric field is measured at 16 positions (also, uniformly distributed around the scatterer) for each incidence. We consider a square scatterer, $d_x = d_y = 0.075\,\text{m}$, divided in four square subdomains characterized by different parameters. The original properties of the scatterer subdomains are $[\tilde{\varepsilon}_\infty^1, \tilde{\varepsilon}_\infty^2, \tilde{\varepsilon}_\infty^3, \tilde{\varepsilon}_\infty^4] = [2,3,4,2]$, $[\Delta\tilde{\varepsilon}^1, \Delta\tilde{\varepsilon}^2, \Delta\tilde{\varepsilon}^3, \Delta\tilde{\varepsilon}^4] = [3,2,2,2]$ and $[\tilde{\tau}^1, \tilde{\tau}^2, \tilde{\tau}^3, \tilde{\tau}^4] = [0.5, 0.5, 1, 1]$ ns. The excitation current density is a modulated Gaussian pulse with central frequency $f_c = 0.5$ GHz and bandwidth 0.5GHz, i.e.

$$J_s = \exp[-(at-4)^2]\sin(2\pi f_c t)u(t) \tag{12}$$

where $a = 5\pi \times 10^8$ sec^{-1}.

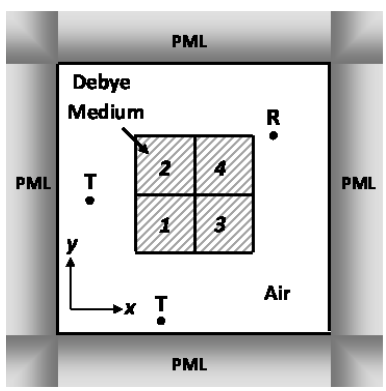

Fig. 1. Geometrical configuration of the 2D Debye dispersive scatterer.

The reconstruction of the scatterer is based on simulated measurements obtained by the FDTD method. The initial estimate of the scatterer properties are $\varepsilon_\infty^{1,2,3,4} = 1.1$, $\Delta\varepsilon^{1,2,3,4} = 0.5$ and $\tau^{1,2,3,4} = 2$ ns. It is assumed that the widths of the four subdomains are *a priori* known for the solution of the inverse problem. Fig 2. shows the cost function versus the number of iterations of the Polak-Ribière algorithm. Moreover, Table 1 shows the original and the estimated values of the scatterer properties, the absolute as well as the relative reconstruction errors of each parameter. After 1500 iterations, the relative reconstruction errors are of order 10^{-2} or lower for all three Debye parameters, depicting the efficiency of the proposed technique.

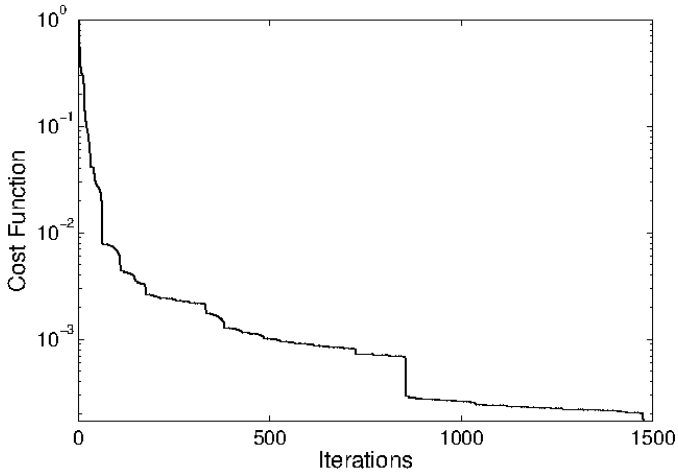

Fig. 2. Cost function vs. the number of iterations of the inversion algorithm.

Table 1. Original and estimated parameters, absolute and relative reconstruction errors.

	$\tilde{x}$	x	$\lvert\tilde{x}-x\rvert$	$\lvert\tilde{x}-x\rvert/\tilde{x}$
ε_∞^1	2.0000	1.9971	0.0029	0.0014
ε_∞^2	3.0000	3.0029	0.0029	0.0010
ε_∞^3	4.0000	4.0012	0.0012	0.0003
ε_∞^4	2.0000	1.9987	0.0013	0.0007
$\Delta\varepsilon^1$	3.0000	2.9784	0.0216	0.0072
$\Delta\varepsilon^2$	2.0000	2.0223	0.0223	0.0112
$\Delta\varepsilon^3$	2.0000	2.0471	0.0471	0.0236
$\Delta\varepsilon^4$	2.0000	1.9503	0.0497	0.0249
τ^1 (ns)	0.5000	0.4954	0.0046	0.0092
τ^2 (ns)	0.5000	0.5071	0.0071	0.0142
τ^3 (ns)	1.0000	1.0239	0.0239	0.0239
τ^4 (ns)	1.0000	0.9747	0.0253	0.0253

5. Conclusions

A time-domain inverse scattering technique for the simultaneous reconstruction of all the characteristic parameters of two-dimensional Debye dispersive media, is proposed. This technique has been based on the calculus of variations. The Fréchet derivatives of the cost function with respect to the scatterer properties are derived analytically. These derivatives can be used by any gradient-based inverse scattering method, whereas the solution of the direct and the adjoint scattering problem can be solved by any time-domain computational technique. In this paper, the FDTD and the Polak-Ribière optimization algorithm have been combined to achieve the reconstruction of all the properties of Debye scatterers. The accuracy of the proposed time-domain inverse scattering method has been depicted by numerical results.

6. References

1. T. Takenaka, H. Jia, and T. Tanaka, “Microwave imaging of electrical property distributions by a forward-backward time-stepping method,” *J. Electromagn. Waves Applicat.*, vol.14, no.12, pp. 1609-1626, 2000.

2. M. Gustafsson and S. He, “An optimization approach to two-dimensional time domain electromagnetic inverse problems,” *Radio Sci.*, vol.35, no.2, pp. 525-536, 2000.

3. I.T. Rekanos and A. Räisänen, “Microwave imaging in the time domain of buried multiple scatterers by using an FDTD-based optimization technique,” *IEEE Trans. on Magn.*, vol.39, no.3, pp. 1381-1384, 2003.

4. S. He, P. Fuks, and G.W. Larson, “An optimization approach to time-domain electromagnetic inverse problem for a stratified dispersive and dissipative slab,” *IEEE Trans. on Antennas Propag.*, vol.44, no.9, pp. 1277-1282, 1996.

5. E. Abenius and B. Strand, “Solving inverse electromagnetic problems using FDTD and gradient-based minimization,” *Int. J. Numer. Methods Eng.*, vol.68, no.6, pp. 650-673, 2006.