Gaussian-Beam Transmission Formula

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Abstract

An exact Gaussian-beam transmission formula is presented for the output of a receiving transducer due to the field of a transmitting transducer. The field of the transmitting transducer is expressed in terms of a Gaussian-beam transmitting characteristic and Gaussian beams propagating in all directions. The output of the receiving transducer is expressed in terms of a Gaussian-beam receiving characteristic and the output of Gaussian-beam receivers. The branch-cut disks for both transmission and reception are centered at the reference point of the transducer. The transmission formula provides an efficient way of computing near-field interactions between transducers.

1 Introduction

Deschamps [1] discovered that an exact solution to Helmholtz equation is obtained if a complex source coordinate is inserted into the expression for the free-space Green's function. This solution is called a complex source point beam and exhibits Gaussian behavior in the paraxial region [2]. We refer to the complex source point beam as simply a "Gaussian beam."

The first exact spherical representation involving Gaussian beams was obtained by Norris [3], who expressed the field of a single real point source at the origin in terms of Gaussian beams emanating from a sphere with complex radius. Heyman [4] derived the analogous time-domain formula using the analytic-signal Fourier transform. Norris and Hansen subsequently generalized these formulas to arbitrary bounded acoustic sources both in the frequency domain [5] and time domain [6]. Also, Huygens' relations that employ Gaussian beams as basis functions have received some attention recently [7], [8].

A recent paper [9] extended the results of [5] to include the exterior source problem where the field is computed inside a sphere that contains no sources. In the present paper we use the results of [9] to derive an exact transmission formula for electro-acoustic transducers solely in terms of Gaussian beams. This formula will complement the standard transmission formulas (based on plane waves, cylindrical waves, and spherical waves) that are widely used to compute the near-field interactions between both transducers and antennas.

The spherical coordinates (r, θ, ϕ) are related to the rectangular coordinates through $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$ with the radial spherical unit vector given by $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\theta\hat{\mathbf{z}}$. Here $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors for the rectangular coordinates (x, y, z), so that $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$. The source-free region of space is an unbounded linear lossless fluid that can be fully described by the pressure $\Phi(\mathbf{r})$. Throughout, k is the wave number, and $e^{-i\omega t}$ time dependence with $\omega > 0$ is assumed and suppressed.

2 Transmitting Transducer

Consider a transmitting transducer with reference point at the origin contained in the sphere $r = D$. The transducer field outside the sphere has the spherical harmonics expansion

$$\Phi(\mathbf{r}) = \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} \mathcal{T}_{\ell m}\, h_\ell^{(1)}(kr) Y_{\ell m}(\theta,\phi), \quad r > D \tag{1}$$

where $\mathcal{T}_{\ell m}$ are spherical-wave transmitting coefficients of the transducer. The truncation limit is $L = kD + \gamma(kD)^{1/3}$ with γ determining the accuracy. In (1), $h_\ell^{(1)}(kr)$ is the spheical Hankel function of the first

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kind [10, Ch. 10], and $Y_{\ell m}(\theta,\phi)$ is the spherical harmoic [11, p.99].

We next invoke an exact formula that was first derived in [5] by solving an integral equation, and more recently derived in [9] through analytic continuation:

$$h_\ell^{(1)}(kr)\,Y_{\ell m}(\theta,\phi) = \int d\Omega'\, G(\mathbf{r}, ia\hat{\mathbf{r}}')\,\frac{Y_{\ell m}(\theta',\phi')}{ik\,j_\ell(ika)}, \quad r > a > 0. \tag{2}$$

Here $\int d\Omega' = \int_0^{2\pi}\int_0^{\pi}\sin\theta'\,d\theta'\,d\phi'$ is an integration over the unit sphere, and

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik\sqrt{(\mathbf{r}-\mathbf{r}')^2}}}{4\pi\sqrt{(\mathbf{r}-\mathbf{r}')^2}} \tag{3}$$

is the free-space Green's function with the square root defined to have non-negative real part and branch cut along the negative real axis. Since $(\mathbf{r}-ia\hat{\mathbf{r}}')^2 = \mathbf{r}\cdot\mathbf{r} - a^2 - 2ia\mathbf{r}\cdot\hat{\mathbf{r}}'$, the branch cut is reached when $\mathbf{r}\cdot\mathbf{r} - a^2 < 0$ and $\mathbf{r}\cdot\hat{\mathbf{r}}' = 0$. Hence, the branch cut manifests itself in real 3D space by the disk of radius a with normal $\hat{\mathbf{r}}'$ centered at the origin.

The wave object $G(\mathbf{r}, ia\hat{\mathbf{r}}')$ is known both as a Gaussian beam and as a complex source-point beam with source point $\mathbf{r}' = ia\hat{\mathbf{r}}'$. It satisfies the homogeneous Helmholtz equation everywhere except on the branch-cut disk where its sources reside [1], [2], [8], [12]. The far-field formula

$$G(\mathbf{r}, ia\hat{\mathbf{r}}') = \frac{e^{ikr}}{4\pi r}\,e^{ka\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'} + O(r^{-2}) \tag{4}$$

shows that for $k > 0$ and $a > 0$ the Gaussian beam radiates most strongly in the direction $\hat{\mathbf{r}}'$ and most weakly in the direction $-\hat{\mathbf{r}}'$. The name Gaussian beam refers to the fact that $G(\mathbf{r}, ia\hat{\mathbf{r}}')$ exhibits Gaussian behavior near the direction $\hat{\mathbf{r}}'$. To see this let $\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}' = \cos\Theta$ and consider small Θ.

The outgoing multipole field $h_\ell^{(1)}(kr)\,Y_{\ell m}(\theta,\phi)$ is thus expressed through (2) in terms of Gaussian beams propagating in all directions. In accordance with (2) the field due to the disk with normal $\hat{\mathbf{r}}'$ is scaled by the factor $Y_{\ell m}(\theta',\phi')$. Inserting the multipole expression (2) into (1) gives

$$\Phi(\mathbf{r}) = \int d\Omega' \mathcal{T}(\hat{\mathbf{r}}')\,G(\mathbf{r}, ia\hat{\mathbf{r}}'), \quad r > D, \quad r > a > 0 \tag{5}$$

where

$$\mathcal{T}(\hat{\mathbf{r}}') = \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} \frac{\mathcal{T}_{\ell m}\,Y_{\ell m}(\theta',\phi')}{ik\,j_\ell(ika)} \tag{6}$$

is the Gaussian-beam transmission characteristic of the transducer. The field of the transducer is expressed in (5) in terms of Gaussian beams propagating in all directions away from the source region. The transducer is thus replaced by sources that reside on a collection of disks centered at the origin with normals pointing in all directions. The Gaussian-beam transmission characteristic $\mathcal{T}(\hat{\mathbf{r}}')$ determines the "strength" of each of these transmitting disks. The formula (5) has the same form as the plane-wave expansion [13, Eq.(6.8)] with the plane waves replaced by Gaussian beams.

3 Receiving Transducer

Consider next a receiving transducer with reference point at the origin contained in the sphere $r = D$. The field due to sources outside the sphere incident upon the transducer can be expressed in terms of standing spherical multipoles as

$$\Phi(\mathbf{r}) = \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} \mathcal{B}_{\ell m}\,j_\ell(kr)Y_{\ell m}(\theta,\phi), \quad r < D. \tag{7}$$

The output V of the transducer can be expressed in terms of the spherical-wave receiving coefficients $\mathcal{R}_{\ell m}$ and the standing-wave spherical expansion coefficients $\mathcal{B}_{\ell m}$ as

$$V = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \mathcal{B}_{\ell m} \mathcal{R}_{\ell m}. \tag{8}$$

In [9] it was shown that the expansion coefficients $\mathcal{B}_{\ell m}$ can be expressed as

$$\mathcal{B}_{\ell m} = \frac{1}{j_\ell(ika)} \int d\Omega\, \Phi(ia\hat{\mathbf{r}})\, Y_{\ell m}^*(\theta,\phi), \quad 0 < a < D \tag{9}$$

where $*$ indicates complex conjugation. Combining (7) and (9) shows that the transducer output is

$$V = \int d\Omega\, \mathcal{R}(\hat{\mathbf{r}})\, \Phi(ia\hat{\mathbf{r}}) \tag{10}$$

where

$$\mathcal{R}(\hat{\mathbf{r}}) = \sum_{\ell=0}^{L} \sum_{m=-\ell}^{\ell} \frac{\mathcal{R}_{\ell m}}{j_\ell(ika)} Y_{\ell m}^*(\theta,\phi) \tag{11}$$

is the Gaussian-beam receiving characteristic of the transducer.

Let us now discuss the quantity $\Phi(ia\hat{\mathbf{r}})$. We can think of it as the output of a point receiver located at the imaginary point $ia\hat{\mathbf{r}}$. To determine the plane-wave receiving characteristic of this complex point receiver, we illuminate it with a plane wave $\Phi(\mathbf{r}) = e^{-ik\hat{\mathbf{r}}^i\cdot\mathbf{r}}$ propagating in the direction $\hat{\mathbf{r}}^i$. The output due to this incident field is simply $\Phi(ia\hat{\mathbf{r}}) = e^{ka\hat{\mathbf{r}}^i\cdot\hat{\mathbf{r}}}$. Comparing with (4) shows $\Phi(ia\hat{\mathbf{r}})$ has the same pattern as a Gaussian beam. Hence, we refer to $\Phi(ia\hat{\mathbf{r}})$ as a Gaussian-beam receiver. The formula (10) thus expresses the output of the transducer as a superposition of Gaussian-beam receiver outputs $\Phi(ia\hat{\mathbf{r}})$ multiplied by the Gaussian-beam receiving characteristic $\mathcal{R}(\hat{\mathbf{r}})$, determined from (11) by the spherical-wave receiving coefficients $\mathcal{R}_{\ell m}$ of the the transducer.

4 Transmission Formula

We next combine the results of the previous two sections to obtain the transmission formula for the output of one transducer due to the field of another transducer (multiple interactions between the transducers are assumed negligible). The transmitting transducer is inside the sphere of radius D centered at $\mathbf{r}_t$, and the receiving transducer is inside the sphere of radius D centered at $\mathbf{r}_r$. The two spheres have no points in common, and the two sets of Gaussian-beam disks do not intersect.

Combining (5) and (10) gives the desired Gaussian-beam transmission formula

$$V = \int d\Omega \int d\Omega'\, \mathcal{T}(\hat{\mathbf{r}}')\, \mathcal{R}(\hat{\mathbf{r}})\, G(\mathbf{r}_r + ia\hat{\mathbf{r}}, \mathbf{r}_t + ia\hat{\mathbf{r}}'). \tag{12}$$

The output of the receiving transducer is expressed here as an integral over two unit spheres where the integrand is the product of three factors: (i) the Gaussian-beam transmitting characteristic of the transmitting transducer, (ii) the Gaussian-beam receiving characteristic of the receiving transducer, and (iii) the free-space Green's function with observation point $\mathbf{r}_r + ia\hat{\mathbf{r}}$ and source point $\mathbf{r}_t + ia\hat{\mathbf{r}}'$. The third factor is simply the output of a Gaussian-beam receiver due to a Gaussian beam.

We have

$$G(\mathbf{r}_r + ia\hat{\mathbf{r}}, \mathbf{r}_t + ia\hat{\mathbf{r}}') = \frac{e^{ik\zeta}}{4\pi\zeta}, \qquad \zeta = \sqrt{(ia\hat{\mathbf{r}} + \mathbf{r}_r - \mathbf{r}_t - ia\hat{\mathbf{r}}')^2} \tag{13}$$

and it can be shown that the imaginary part of ζ has the bounds [9]

$$-2a \leq \operatorname{Im}(\zeta) \leq 2a. \tag{14}$$

The lower bound is attained only when the transmitting and receiving disk normals point toward each other, and the upper bound is attained only when the disk normals point away from each other. Even for moderate values of ka, the magnitude of $e^{ik\zeta}$ attains values in a wide interval. Hence, we can omit large portions of the unit spheres for the transmitting and receiving disk normals without introducing appreciable errors in the computation of the transducer output. The analysis in [9] of the numerical properties of a Gaussian-beam field-computation formula carries over to the transmission formula (12).

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