Error Analysis of the Gaussian Beam Summation Method for Ultra Wide-Band Radiation in Inhomogeneous Medium

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Abstract

We address issues related to the accuracy of the beam summation method (BSM) for ultra wide-band (UWB) radiation in inhomogeneous medium. In the BSM, the field is expanded using a lattice of beam propagators that are tracked locally in the medium. As a test bed for the accuracy of the BSM, we derive a solution for a beam in a general plane stratified medium which is valid beyond the paraxial regime. We then use it to derive bounds on the error away from the beam axis, and to obtain rules for optimizing the choice of the parameters in the BSM.

1. Introduction

The BSM is an important tool in wave theory, since it provides a systematic framework for ray-based construction of spectrally uniform local solutions in complex configurations [1–7]. In these formulations, the field is expanded into a phase-space spectrum of beams that emanate at a given set of points and directions in the source domain, and thereafter are tracked locally in the medium. The advantages of the beam formulations over the more traditional representations are: 1) unlike the plane waves, the beam propagators can be tracked locally in inhomogeneous media or through interactions with interfaces, and unlike rays, they are insensitive to the geometrical optics (GO) transition zones; 2) the formulations are a priori localized around the phase-space skeleton of GO (the so-called Lagrange manifold) since only those beam propagators that pass near the observation point actually contribute there. Thus, BSM combines the algorithmical ease of GO with the uniform features of spectral representations, and therefore have been used recently in various applications. The parameters of the expansion beams (e.g., width and collimation) can be chosen by the wave-modeler to obtain effective representations for the field in complex configurations.

A discrete phase space BSM representation for UWB radiation from extended source distributions was introduced in [4]. In this formulation, the aperture field is described by beams that emerge along a discrete lattice of initiation points and orientations in the source domain. What distinguishes the UWB-BSM of [4] is the use a *frequency independent* lattice of beam initiation points and orientations, and the use of Iso-diffracting Gaussian beams (ID-GB) whose propagation characteristics are described by *frequency independent* parameters. Consequently, the calculation of the ID-GB propagation in the ambient environment needs to be done only once and then used for all the relevant frequencies in the source. In view of these features, this BSM representation has also been formulated directly in the space-time domain [5]. UWB-BSM formulations for point sources in the frequency and in the time domains have been given in [6, 7].

As mentioned above, the BSM consists of two phases: 1) expansion of the source distribution into beams; 2) tracking the propagators in the ambient environment. This paper addresses only the latter and explores the validity of the known closed form paraxial solutions for ID-GB propagation in general inhomogeneous media [9]. As a test bed to validate these solutions, we consider a general plane stratified medium and derive solutions that are valid beyond the paraxial regime. Such solutions can be derived via the CS approach, but this requires an extension of the inhomogeneous wave-speed to the complex coordinates space. To circumvent this difficulty, we use a generalized plane-wave representation in which the CS field is used as the initial conditions, and derive closed form solutions via an asymptotic evaluation of the spectral integrals. Using these solutions we obtain the desired expressions for the error of the paraxial GB solution, and conclude some optimization rules for the choice of the beam parameters in the BSM. Specifically, we study the error near regions of phenomenological interest such as caustics.

The BSM contains some free parameters (e.g., the beam collimation length b) that should be properly chosen by the wave-modeler in order to reduce the error and the computational complexity (e.g., the number of beams needed to calculate each field point). The present conclusions regarding the

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error of a single beam as a function of the parameters are used to define a set of optimization rules for choosing the BSM parameters.

Finally, we should mention in that context the pragmatic BSM algorithm in [6] for tracking source excited fields in plane stratified medium. In that algorithm, the medium is approximated by layers with constant gradient of the wave speed, so that the GB propagators can be calculated recursively in a closed form. This approach is significantly more accurate and efficient than the standard stair case approximation, or the WKB-type solution. Using our proposed solution, we study the error of the constant gradient approximation and derive conditions for the optimal meshing of the wave speed.

2. Spectral Representation for Gaussian Beams Beyond the Paraxial Region

Referring to Fig. 1, we consider a Gaussian-like beam propagating in 3D space $\mathbf{r} = (x, y, z)$ in a smooth plane stratified medium, identified by the wave-speed $v(z)$. For convenience, we define the refractive index $n(z) = v_0/v(z)$, where v_0 is the reference wave-speed in the $z = 0$ plane. For the sake of discussion only, $n(z)$ is taken to be monotonically decreasing with z. We assume a weakly inhomogeneous medium with characteristic length $L(z) \triangleq n/|n'| \gg \lambda$, where $n' = \partial_z n$ and $\lambda(z)$ is the local wave-length. Without loss of generality, the beam emerges from $\mathbf{r}_0 = (0, 0, 0)$ at an angle γ_0^i relative to the z-axis and propagates along the ray trajectory Σ in the xz plane (see Fig. 1). The beam can be modeled by using a point source located at the complex coordinates point [3]: $\mathbf{r}' = \mathbf{r}_0 + i\mathbf{b}$, where $\mathbf{b} = b(\sin\gamma_0^i, 0, \cos\gamma_0^i)$ with $b > 0$. In a uniform medium, one obtains an exact closed-form expression for the complex source beam (CSB) that behaves paraxially like a GB propagating in the $\mathbf{b}$ direction with Fresnel length $b = |\mathbf{b}|$ and a waist at $\mathbf{r} = \mathbf{r}_0$. However, as mentioned in the introduction, the CS concept is hard to implement in inhomogeneous medium, hence we use here a spectral representation with the CSB field as the initial condition in the $z = 0$ plane. Addressing first the part of the beam before the turning point (Fig. 1), the generalized plane-wave representation for the beam field $U(\mathbf{r})$ can be cast into the generic form

$$U(\mathbf{r}) = (-i\omega)\frac{i}{2\pi}\int_{\mathcal{C}_\eta}\int_{\mathcal{C}_\xi} d\xi d\eta A(\mathbf{r};\xi,\eta)e^{ik_0\Psi_t(\mathbf{r};\xi,\eta)}, \tag{1}$$

where $k_0 = \omega/v_0$ and a $e^{-i\omega t}$ time-dependence is assumed and suppressed. The spectral variables ξ and η are frequency independent and have a pure geometrical interpretation. The amplitude and phase of the plane-wave constituents A and Ψ in (1) are found via the WKB approximation to be [8]

$$A(\mathbf{r};\xi,\eta) = A_0\sqrt{\zeta_0(\xi,\eta)/\zeta(z;\xi,\eta)}, \quad A_0 = -ib/4\pi v_0\zeta_0, \tag{2}$$

$$\Psi_t(\mathbf{r};\xi,\eta) = \Psi_0 + \xi x + \eta y + \int_0^z \zeta(z';\xi,\eta)dz', \quad \Psi_0(\xi,\eta) = +ib\big(1 - \xi\sin\gamma_0^i - \zeta_0\cos\gamma_0^i\big), \tag{3}$$

where $\zeta(z;\xi,\eta) = \sqrt{n^2(z) - \xi^2 - \eta^2}$ is the local spectral variable in the z direction with $\mathrm{Re}\zeta \geq 0$ and $\zeta_0 = \zeta|_{z=0}$. It can also be written as $\zeta = n(z)\cos\gamma(z)$, where $\gamma(z)$ is the local angle with respect to the z-axis, satisfying Snell's law $n(z)\sin\gamma(z) = \sin\gamma_0 = (\xi^2 + \eta^2)^{1/2}$ with $\gamma_0 = \gamma|_{z=0}$. Ψ_0 and A_0 are the phase and amplitude of the initial spectrum at the $z = 0$ plane due to a CS at $\mathbf{r}' = ib(\sin\gamma_0^i, 0, \cos\gamma_0^i)$. Expanding Ψ_0 to second order, one finds that the initial spectrum is Gaussian about $(\xi, \eta) = (\xi^i, 0)$, $\xi^i = \sin\gamma_0^i$,

$$k_0\Psi_0 \approx i\big[(\xi - \xi^i)^2/\theta_d^2\cos^2\gamma_0^i + \eta^2/\theta_d^2\big], \qquad \theta_d = (bk_0)^{-1/2}. \tag{4}$$

θ_d is the beam diffraction angle with collimated beams described by $\theta_d \ll 1$. In what follows, we shall use the local beam coordinates (σ, x_1, x_2) (see Fig.1) where σ is the arc length measured along the beam axis Σ and x_j, $j = 1, 2$, are the transversal distances from the beam axis in the xz and yz planes, respectively. The beam has a waist at $\mathbf{r}_0$ whose $e^{-1/2}$ widths along x_j are $W_0 = (b/k_0)^{1/2}$. Finally, we note that turning beams can be also treated by replacing the integral bounds in (3) to correspond to the turning trajectory and by adding a factor of $-i$ to A in (2).

3. Analysis of the Error of the Paraxial Solution

As mentioned in the Introduction, the GB solution is fully described by solving a set of complex Hamilton-Jacobi equations along the propagation trajectory [9]. This approach is also termed "dynamic ray tracing." In order to explore the error of this approximation, we resort to an alternative route and derive the GB solution from the spectral integral (1). The GB is then obtained by approximating this integral to second order around the spectral center of mass $(\xi, \eta) = (\xi^i, 0)$ and the beam axis $(x_1, x_2) = (0, 0)$, and evaluating the resulting Gaussian integral exactly. The error of this approximation can then be estimated by retaining the higher order terms in (1) and evaluating the resulting integral via a 2D saddle point approach. We use the normalized coordinates $\bar{x}_j = x_j/W_j$, $j = 1, 2$, where the beam widths are given by $W_j(\sigma) = \theta_d\sqrt{S_j^2 + b^2 P_j^2}$ and $S_j(\sigma)$ and $P_j(\sigma)$ are the geometrical ray-spreads in the j^{th} principle plane for point-source and plane-wave excitations, respectively. Specifically, we explore the error along the beam axis as a function of $\bar{x}_1$ for $\bar{x}_2 = 0$, and as a function of $\bar{x}_2$ for $\bar{x}_1 = 0$. The former is found to be

$$\epsilon_1(\sigma, \bar{x}_1) \triangleq \big|U_{\mathrm{H}} - U_{GB}\big| / \big|U_{GB}(\mathbf{x} = 0)\big| \sim \theta_d\big[c_1(\sigma; b)\bar{x}_1^3 + c_2(\sigma; b)\bar{x}_1\big]e^{-\bar{x}_1^2/2}, \tag{5}$$

where θ_d is given in (4) while $c_{1,2}$ are slowly varying functions of σ (to be specified in the presentation). The error ϵ_2 along the x_j axis can be calculated in a similar fashion, but for the case of oblique propagation along the $j = 1$ axis considered here, this error is $\sim O(\theta_d^2)$ and therefore can be neglected compared with ϵ_1.

Figs. 2–4 explore an example for a GB propagating in a caustic forming medium defined by $n(z) = \sqrt{1 - z/L_0}$ where L_0 is a constant. The ray system and the caustic for a point source configuration are shown in Fig. 2. Fig. 3 depicts the error ϵ_1 as a function of b at the caustic. It is seen that the optimal value in this case is near $b = 3L$. Fig. 4 depicts the error vs. x_1.

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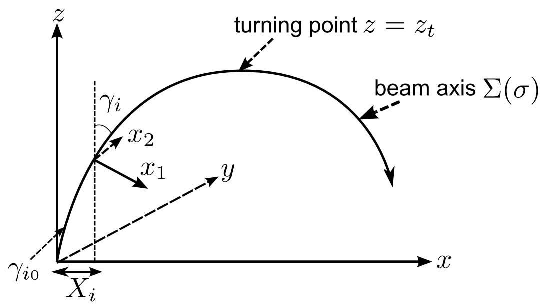

Figure 1: A GB propagating along the ray trajectory Σ emerging from $\mathbf{r}_0 = (0, 0, 0)$ at an angle γ_0^i in the xz-plane. The local coordinate frame is (σ, x_1, x_2) where σ is the arc length along Σ and $x_{1,2}$ are the normal coordinates with respect to Σ in the principle planes xz and yz, respectively.

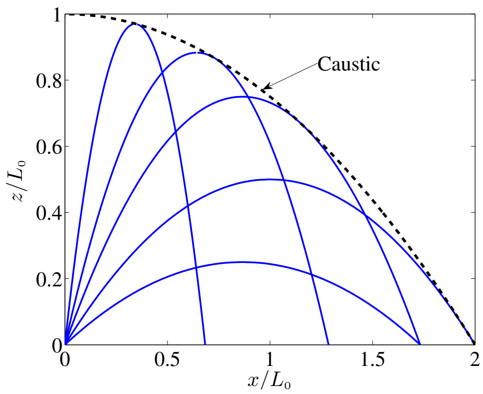

Figure 2: Ray trajectories for point source excitation in a plane stratified medium with refractive index $n(z) = \sqrt{1 - z/L_0}$, with $L_0/\lambda_0 = 10^6$ and $\lambda_0 = \lambda(z = 0)$. The caustic is indicated by a black dashed line.

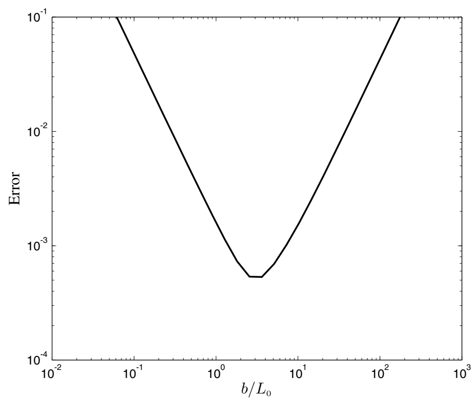

Figure 3: Off-axis error ϵ_1 of the GB near the caustic of Fig. 2 at $x_1 = W_1$ as a function of b.

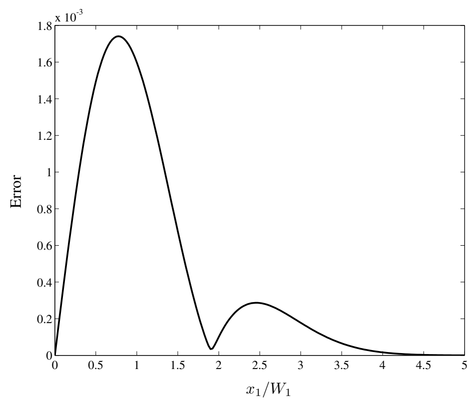

Figure 4: Off-axis error ϵ_1 of the GB as a function of $\bar{x}_1$ near the point where the beam axis touches the caustic of Fig. 2. The beam collimation parameter is $b = L_0$.