Hierarchical and Conforming Nédélec Elements for Surface and Volumetric Cells

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Abstract

New families of hierarchical curl and divergence-conforming vector bases for the most commonly used two- and three-dimensional cells are directly constructed from orthogonal scalar polynomials to enhance their linear independence, which is a simpler process than an orthogonalization applied to the final vector functions. These functions span the mixed-order (or reduced) spaces of Nédélec and can be used to deal with structures meshed by a mixture of cells of different geometry.

1. Introduction

Hierarchical vector basis functions for surface or volumetric representations of electromagnetics fields became increasingly popular in Computational Electromagnetics (CEM) applications since the early 1990s because they facilitate adaptive refinement procedures that do not require any re-meshing of the structures under study. A set of vector basis functions is referred to as hierarchical if the vector basis functions forming the n-th-order set are a subset of the vector basis functions forming the $(n+1)$-th-order set. With these basis functions, the polynomial order of the vector-field solutions can be arbitrarily selected (or adjusted) in different regions of the computational domain to be higher in the regions where, for example, either the energy of the solution or the solution errors are expected (or found) to be higher. In this connection we should mention that the polynomial order of the used vector basis functions can be locally changed also while using interpolatory vector bases, although with a more complex procedure that results to be unsuitable for adaptive p-refinement. In fact, in adaptive p-refinement, the polynomial order of the basis functions is locally increased during an iterative solution process on the same given mesh. Conversely, the adaptation technique is called h-adaption or h-refinement when performed automatically by refining the mesh in the regions where the desired accuracy is not obtained. The main issue in h-refinement is that the refine mesh may contain undesirable features such as ill-shaped elements, while the main user's drawback is that h-refinement always requires a full mastering of the mesher-code and database.

In contrast to interpolatory vector bases [1, 2], hierarchical bases often exhibit poor linear independence as the order of the representation is increased, resulting in an ill-conditioned system of discretized equations which stands out against the use of hierarchical basis functions of very high order, to the point to make h-adaptation unavoidable in several applications. The loss of linear independence is often mitigated by resorting to some cumbersome (partial) orthogonalization process that involves all the used basis vectors. In order to alleviate the loss of linear independence, we have recently proposed new families of hierarchical vector bases for the most commonly used two- and three-dimensional cells [3]-[7] that are directly constructed from orthogonal scalar polynomials, which is a simpler process than an orthogonalization applied to the final vector functions [8]. We considered both *curl* and *divergence-conforming* bases with continuous tangential or normal components, respectively, across adjacent elements. In practice, curl-conforming basis functions are appropriate for discretizations of the vector Helmholtz operator, while divergence-conforming functions are appropriate for the so-called electric field integral equation (EFIE) operator. In contrast to scalar bases, conforming vector bases not only provide appropriate tangential or normal continuity at element (material) interfaces, but also provide convenient means for imposing boundary conditions on unknown fields or currents. They also avoid the spurious modes usually encountered when scalar representations are used with one of the foregoing operator equations.

The existing basis functions can be classified into three groups [3]: A) those that span complete polynomial vector spaces, B) those that span the mixed-order spaces of Nédélec [9] (sometimes known as reduced gradient spaces for

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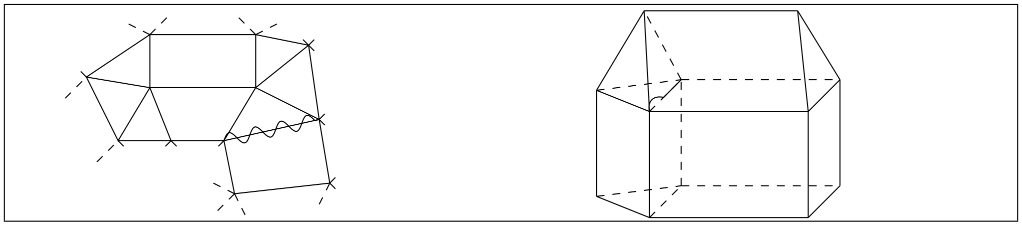


Figure 1: Two and three-dimensional structures meshed by cells of different geometry. At left, the figure shows a 2D-structure meshed by a mixture of triangular and quadrilateral cells while, at right, the figure shows a 3D structure meshed by a mixture of tetrahedral, brick and prismatic cells.

curl-conforming functions), and C) those with subsets that exactly span both types of spaces. Our new hierarchical basis sets belong to group B. Our functions incorporate scale factors that help optimize the resulting matrix conditioning, as discussed in [10], [11].

Our hierarchical vector bases are obtained by a three-step process. For each element, we first define a complete set of generating *scalar* polynomials orthogonal on the given parent element, and therefore hierarchical *in nature*. These polynomials are then multiplied by the zeroth-order vector functions for the element under consideration to obtain a set of vector functions that contains some functions linearly dependent on the others. Finally, by using a procedure similar to that discussed in [1, 2] for the interpolatory case, any redundant basis function is eliminated from the resulting vector set. After removing the redundant vector functions, the total number of the new hierarchical vector basis functions is equal to the number of the equivalent interpolatory basis set given in [1, 2], and the space spanned by the hierarchical and the interpolatory bases is exactly the same.

2. Hierarchical Curl-Conforming Bases

Our new hierarchical curl-conforming basis functions for triangular, quadrilateral, tetrahedral, brick and prism cells are described in [3]-[6]. To guarantee tangential continuity across adjacent cells of different geometry (see Fig. 1), all our curl-conforming vector basis functions are subdivided from the outset into three different groups of edge, face, and volume-based functions and, in each group, all the generating polynomials are mutually orthogonal independent of the definition domain of the inner product, *i.e.* either the volume, the face, or the edge of the cell at issue. Furthermore, to the same purpose, our hierarchical edge and face-based generating polynomials are obtained to be either symmetric or antisymmetric with respect to the parent variables that describe each edge and face of the cell. The tangential continuity of the field across element boundaries is thus simply ensured by adjusting the vector basis function sign to correspond to an arbitrarily selected reference direction along the adjacent elements (see [3]-[6]).

The relative performance of various hierarchical curl-conforming basis functions for triangular cells was evaluated in [10] by discretizing the vector Helmholtz equation

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla} \times \boldsymbol{H} = k^2 \boldsymbol{H} \tag{1}$$

to calculate the resonant frequencies of two-dimensional cavities bounded by perfectly conducting walls (a homogeneous Neumann boundary). When the magnetic field $\boldsymbol{H} = \sum_i \alpha_i \boldsymbol{B}_i$ is expressed in terms of vector basis functions $\boldsymbol{B}_i$ the element matrices $\mathbf{S}$ and $\mathbf{T}$ have entries of the form

$$\mathrm{S}_{mn} = \iint_S \boldsymbol{\nabla} \times \boldsymbol{B}_m \cdot \boldsymbol{\nabla} \times \boldsymbol{B}_n \, \mathrm{d}S \tag{2}$$

$$\mathrm{T}_{mn} = \iint_S \boldsymbol{B}_m \cdot \boldsymbol{B}_n \, \mathrm{d}S \tag{3}$$

Because of the nullspace of the curl operator, the element matrix $\mathbf{S}$ is singular. However, $\mathbf{T}$ is nonsingular and its condition number provides an indication of the relative linear independence of the different basis functions [10].

TABLE I: CONDITION NUMBER COMPARISON FOR TRIANGULAR BASES OF ORDER 2.5 AND 3.5

The matrix condition number of the global **T**-matrix in (3) obtained by taking the ratio of the largest singular value to the smallest, for five triangular-cell meshes of different quality. The Graglia *et al.* bases are in their original form [3]; the Ingleström bases are scaled using the optimal factors of [10].

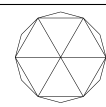
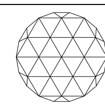
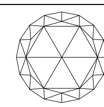
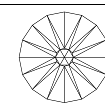
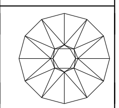
Family	Mesh 12	Mesh 42	Mesh 40a	Mesh 40b	Mesh 34
Graglia, Wilton, and Peterson (GWP) [1] of order 2.5	528	352	597	6,075	3,319
Graglia* *et al.* [3] of order 2.5	720	459	855	1,923	1,366
Ingleström [12] of order 2.5	1,040	510	1,085	5,022	3,284
GWP [1] of order 3.5	2,274	1,541	2,689	58,813	20,857
Graglia *et al.* [3] of order 3.5	2,298	1,502	3,160	6,449	5,093
Ingleström [12] of order 3.5	1,461	1,148	2,526	14,002	7,162
					
Number of cells:	12	42	40	40	34
Order of **T** for 2.5 order:	114	468	444	444	384
Order of **T** for 3.5 order:	240	792	752	752	648

Table I shows a comparison of the condition numbers of the global **T**-matrix obtained from five different triangular-cell meshes for the functions of Nédélec order 2.5 and 3.5. (The mesh numbers indicate the number of cells.) Mesh #40b and mesh #34 were deliberately designed to have cells with a large aspect ratio, resulting in a poor matrix condition number. A singular value decomposition algorithm was used to find the largest and smallest singular values of the global **T**-matrix, and the ratio of those two parameters is reported. In the Table, the interpolatory basis functions of [1], with unit components, are used as a reference and the results obtained for our new hierarchical functions are compared with those obtained by using the Ingleström functions [12] which, among all the other existing bases, are those able to provide the lowest condition numbers if properly scaled as discussed in [10]. For most of the other hierarchical families, the use of "optimal" scale factors substantially improves the resulting matrix condition numbers whereas the condition numbers associated with our new family improved slightly by optimizing the scale factors, that is our functions appear to be scaled in a fairly optimal manner already. Table II presents element matrix condition numbers for vector bases of increasing order on square and cubic reference cells of unitary edge-length. Our proposed hierarchical bases are compared to the interpolatory bases of [1]. The hierarchical condition numbers clearly grow at a much slower rate as their order increases than those of the interpolatory set. Preliminary numerical results (not shown here) were also reported for skewed quadrilateral-cell bases in [11]. These results showed that of the other existing families, only the functions of [13] were comparable to the condition numbers of our new set; the other families exhibited higher condition numbers. Preliminary results obtained by use of hierarchical curl-conforming vector bases are reported in [6].

3. Hierarchical Divergence-Conforming Bases

Divergence-conforming functions on triangles or quadrilaterals are easily obtained by a 90 degree rotation of the curl-conforming bases. New hierarchical divergence-conforming basis functions for the most commonly used three-dimensional cells are described in [7]. To guarantee the element conformity when using cells of different geometry, all our divergence-conforming vector basis functions are subdivided from the outset into two different groups of face and volume-based functions and, in each group, all the generating polynomials are mutually orthogonal independent of the definition domain of the inner product, *i.e.* either the volume or the face of the cell at issue. The normal continuity of the field across element boundaries is simply ensured by adjusting the basis function sign to correspond to an arbitrarily selected reference direction along the adjacent elements (see [7]).

TABLE II: INDIVIDUAL ELEMENT **T**-MATRIX CONDITION NUMBERS FOR SQUARE AND CUBIC CELLS

Basis order	Order of **T**	CNH	CNI	Order of **T**	CNH	CNI
0.5	4	3.000	3	12	9	9
1.5	12	22.956	20.639	54	526.998	141.988
2.5	24	22.956	109.720	144	526.998	1.02709×10^3
3.5	40	60.483	486.676	300	3.65825×10^3	6.88114×10^3
4.5	60	60.483	2.5579×10^3	540	3.65825×10^3	6.04903×10^4
5.5	84	114.658	1.60285×10^4	882	1.31464×10^4	7.08665×10^5

Individual element **T**-matrix condition numbers for hierarchical (CNH) and interpolatory (CNI) [1] *vector bases* of different order, obtained by considering a square (left-hand Table) and a cubic (right-hand Table) cell of unitary edge-length.

4. Conclusion

Above, the most important features of our new hierarchical vector bases which span the mixed-order spaces of Nédélec have been considered. References [3]-[7] present the construction details of the curl-conforming bases; in 2D the divergence-conforming bases on triangular or quadrilateral cells can be obtained by a 90 degree rotation of the curl-conforming bases. The presentation will include a detailed discussion of the construction technique used for hierarchical divergence-conforming bases on volumetric cells.

4. References

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