On the quality factor of a low-loss parallel-plate resonator based on complementary metamaterials

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Abstract

Several proposals have been made during the last ten years on size-independent resonators by using metamaterials for phase compensation. In this paper, we present an estimate of the quality factor of a simple parallel-plate resonator, based on realistic assumptions such as linear, passive, causal, time-translational invariant metamaterials, and lossy walls. It is found that the quality factor of the resonator depends on whether the losses of the material or the metal walls are dominating: in the former case, there is no size dependence, in the latter case, the size dependence is similar to a vacuum-filled resonator.

1 Introduction

Several proposals on size-independent resonators have been made the last ten years [1–4]. The basic idea is that the phase built up within a "standard" material (positive refractive index) is cancelled by the negative phase in a complementary metamaterial (negative refractive index). The total phase change in the material is then zero, for all frequencies where the complementary nature of the constituents can be maintained.

The challenge of the design is the dispersive nature of any material experiencing negative refractive index. It is shown in [5] that a specific value of the refractive index can not be maintained to an accuracy larger than the fractional bandwidth, if this value is less than the high frequency response. In this paper, we show that this result applies also to the archetypical size-independent parallel-plate resonator proposed by Engheta [1].

The results in [5] are based on four simple physical principles: linearity, passivity, causality, and time translational invariance. Thus, they are independent of the actual realization of the materials as long as these principles are satisfied, which is very important for metamaterials that are often made up as composite structures involving complex geometries [6, 7].

2 The parallel-plate resonator with two materials

The original resonator proposed in [1] is a one-dimensional parallel plate geometry, see Figure 1. The metal plates are separated by a distance $d = d_1 + d_2$, where each region $-d_1 < z < 0$ and $0 < z < d_2$ is filled with materials with refractive indices n_1 and n_2, respectively. Assuming time dependence e^{st} and the polarization $\boldsymbol{E} = E\hat{\boldsymbol{x}}$ and $\boldsymbol{H} = H\hat{\boldsymbol{y}}$, the fields in each region are

$$E_1(z) = E_{1+}\mathrm{e}^{-sn_1(s)z/c_0} + E_{1-}\mathrm{e}^{sn_1(s)z/c_0} \qquad -d_1 < z < 0 \qquad (1)$$

$$H_1(z) = \frac{1}{\eta_1}E_{1+}\mathrm{e}^{-sn_1(s)z/c_0} - \frac{1}{\eta_1}E_{1-}\mathrm{e}^{sn_1(s)z} \qquad -d_1 < z < 0 \qquad (2)$$

$$E_2(z) = E_{2+}\mathrm{e}^{-sn_2(s)z/c_0} + E_{2-}\mathrm{e}^{sn_2(s)z/c_0} \qquad 0 < z < d_2 \qquad (3)$$

$$H_2(z) = \frac{1}{\eta_2}E_{2+}\mathrm{e}^{-sn_2(s)z/c_0} - \frac{1}{\eta_2}E_{2-}\mathrm{e}^{sn_2(s)z/c_0} \qquad 0 < z < d_2 \qquad (4)$$

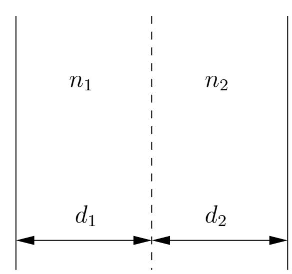

Figure 1: Geometry of the parallel plate resonator.

The wave impedances η_1 and η_2 can be chosen as $\eta_1 = \eta_2 = \eta_0 \approx 377\,\Omega$ without violating any of the four physical principles of linearity, passivity, causality, or time translational invariance. Hence, this choice is

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made for simplicity of presentation. Adding the boundary conditions of continuity at $z = 0$, $E_1(0) = E_2(0)$ and $H_1(0) = H_2(0)$, and a surface impedance Z at each wall, $E_1(-d_1) = -ZH_1(-d_1)$ and $E_2(d_2) = ZH_2(d_2)$, results in the resonance condition

$$\mathrm{e}^{2s(d_1n_1(s)+d_2n_2(s))/\mathrm{c}_0} = \mathrm{e}^{2sdn(s)/\mathrm{c}_0} = \left(\frac{1 - Z(s)/\eta_0}{1 + Z(s)/\eta_0}\right)^2 \tag{5}$$

where we introduced the total refractive index $n(s) = (d_1n_1(s)+d_2n_2(s))/d$. By our four physical principles, both $sn_1(s)$ and $sn_2(s)$ are guaranteed to be positive real functions, *i.e.*, analytic mappings of the right half complex plane to itself, mapping the positive real line to itself. Since the function $sn(s)$ is a convex combination of two positive real functions, it is itself a positive real function. The surface impedance of a highly conducting metal wall ($\sigma \approx 10^7\,\mathrm{S/m}$) is

$$Z(s) = \sqrt{\frac{s\mu}{\sigma}} = \eta_0\sqrt{s\tau}, \quad \tau = \epsilon_0/\sigma \approx 10^{-18}\,\mathrm{s} \tag{6}$$

Thus, $Z(s)/\eta_0$ is a very small number up to frequencies approaching THz, where the conductivity model of most metals needs to be refined. Taking this into account, the resonance condition (5) can be written to first order in $Z(s)/\eta_0$ as

$$\mathrm{e}^{2sdn(s)/\mathrm{c}_0} = 1 - 4Z(s)/\eta_0, \quad \Rightarrow \quad sdn(s)/\mathrm{c}_0 = m\mathrm{j}\pi - 2Z(s)/\eta_0, \quad m = 0, \pm1, \pm2, \ldots \tag{7}$$

The standard vacuum-filled half-wave resonator is given by $n_1 = n_2 = n = 1$ and $m = 1$. The resonance frequency for this mode is $\omega_0 = \pi\mathrm{c}_0/d$, and upon assuming nonmagnetic metal walls the Q factor can be estimated as [8, p. 116]

$$Q = \frac{\pi\eta_0}{4R_\mathrm{s}} = \frac{\pi\sqrt{2}}{4\sqrt{\omega_0\tau}} = \frac{d}{2\delta} \tag{8}$$

where $R_\mathrm{s} = \sqrt{\mu_0\omega_0/(2\sigma)}$ is the surface resistance of the metal at the resonance frequency, and $\delta = \sqrt{2/(\mu_0\sigma\omega_0)}$ is the skin depth of the metal walls. This qualitative result applies to cavity resonators of general shape as well, where $Q = CV/(S\delta)$, with C being a geometrical factor close to unity, V is the volume of the cavity, and S is the surface area of the walls [9, p. 373].

The idea of the size-independent resonator is to consider the choice $m = 0$, which requires $n(s) \approx 0$ for non-zero s, *i.e.*, one of the materials needs to have a negative refractive index. Taking a small surface impedance into account, the resonance condition for the zeroth order mode can be written

$$sdn(s)/\mathrm{c}_0 = -2Z(s)/\eta_0 = -2\sqrt{s\tau} \tag{9}$$

Even though we cannot solve for the resonance frequency s_0 without an explicit model for $n(s)$, we can draw some qualitative conclusions. Since $sn(s)$ and $Z(s)$ are both positive real functions, there can be no solution s in the right half plane. This corresponds to the passivity requirement that the fields do not grow exponentially in time. Since the square root function (with branch cut along the negative real axis) maps the whole complex plane to the right half plane, we also see that $sn(s)$ must be in the left half plane at the resonance frequency $s = s_0$, which itself must be in the left half plane if it exists. To draw conclusions on the quality factor or bandwidth of the size-independent resonator, we need other means, which are provided by [5].

3 Sum rule

The resonance condition (7) suggests that for the zeroth order resonance $m = 0$, we should study when the positive real function $sdn(s)/\mathrm{c}_0 \approx 0$. We assume it has high frequency asymptotic $sdn_\infty/\mathrm{c}_0 + \mathrm{o}(s)$ as $s\hat{\to}\infty$, where the notation $\hat{\to}$ indicates limits inside the right half plane. Due to the definition of $n(s)$ in (5), the high frequency response $n_\infty = (d_1n_{1\infty} + d_2n_{2\infty})/d$ is a convex combination of the individual responses

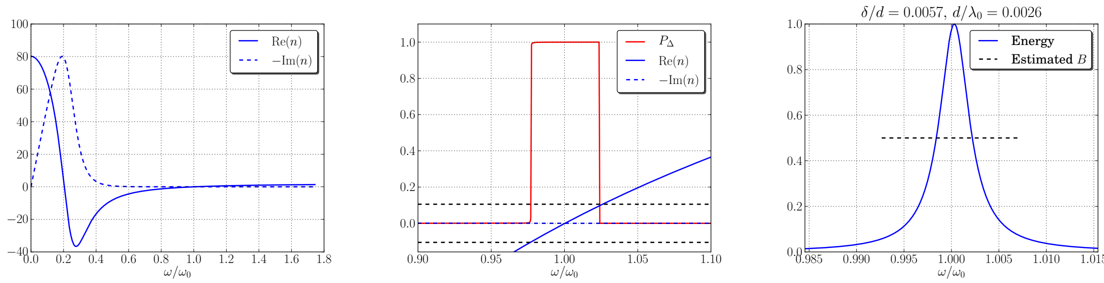

Figure 2: Left: Refractive index $n(\mathrm{j}\omega) = 1 + n_{\mathrm{m}}(\mathrm{j}\omega)$, where $n_{\mathrm{m}}(\mathrm{j}\omega)$ is given explicitly in [10] as an example of a refractive index which is close to -1 with small imaginary part. Middle: Illustration of sum rule (10), the black dashed lines correspond to $\varDelta$ in (11). Right: Frequency distribution of energy in the resonator and bandwidth estimate (16).

$n_{1\infty}$ and $n_{2\infty}$, and hence $\min(n_{1\infty}, n_{2\infty}) \leq n_\infty \leq \max(n_{1\infty}, n_{2\infty})$, and $n_\infty \geq 1$ due to special relativity. In [5] the following sum rule was derived for functions with this kind of asymptotics

$$\int_0^\infty \frac{1}{\pi} \arg\left(\frac{\frac{\omega}{\omega_0} n(\mathrm{j}\omega) - \varDelta}{\frac{\omega}{\omega_0} n(\mathrm{j}\omega) + \varDelta}\right) \frac{\mathrm{d}\omega}{\omega_0} = \frac{\varDelta}{n_\infty} \tag{10}$$

for any ω_0 and $\varDelta$. Here, we choose the angular frequency ω_0 as the zero order resonator frequency where $n(\mathrm{j}\omega_0) \approx 0$, and $\varDelta$ is a measure of how closely this condition is satisfied. This sum rule is well suited to study when $n(\mathrm{j}\omega) \approx 0$, since that is the only region where the integrand differs significantly from zero. More explicitly, $P_\varDelta(z) = \frac{1}{\pi} \arg\left(\frac{z-\varDelta}{z+\varDelta}\right)$ has the properties $P_\varDelta(z) \leq 1$ for all z, $P_\varDelta(z) \geq 1/2$ for $|z| < \varDelta$, $P_\varDelta(z) = 1$ for $|z| < \varDelta$ and $\operatorname{Im} z = 0$, and finally $P_\varDelta(z) = 0$ for $|z| > \varDelta$ and $\operatorname{Im} z = 0$. Thus, for $\operatorname{Im} z = 0$, the function $P_\varDelta(z)$ is an indicator function of the condition $|z| < \varDelta$, *i.e.*, it is one where the condition is satisfied and zero elsewhere, see Figure 2. In terms of the refractive index n this condition is

$$|n(\mathrm{j}\omega)| \leq \frac{\omega_0 \varDelta}{\omega} \tag{11}$$

Denote the fractional bandwidth where this condition is satisfied by $B_\varDelta$. The sum rule (10) then implies the following physical bound for low loss materials [5]

$$\frac{B_\varDelta}{1 + B_\varDelta/2} \leq \frac{\varDelta}{n_\infty} \quad \Rightarrow \quad B_\varDelta \leq \frac{\varDelta/n_\infty}{1 - \varDelta/(2n_\infty)} \tag{12}$$

where $\varDelta$ is an arbitrary number. Note that this bound is only concerned with the materials, based on their nature as passive, causal, linear systems, and no connection is yet made with the actual application as a resonator.

Consider the full solution as the result of one wave with amplitude E_0 being reflected an infinite number of times, where the mth reflected wave has amplitude $(\varGamma(\mathrm{j}\omega)^2 \mathrm{e}^{-2\mathrm{j}\omega n(\mathrm{j}\omega)d/\mathrm{c}_0})^m E_0$. Then the total field is proportional to the factor $E_0/(1 - \exp(-2\mathrm{j}\omega n(\mathrm{j}\omega)d/\mathrm{c}_0)\varGamma(\mathrm{j}\omega)^2)$ after summing the geometric series. The half-maximum full width $B_{1/2}\omega_0 = \omega_0/Q$ of the resonator is the bandwidth for the condition

$$\left|1 - \mathrm{e}^{-2\mathrm{j}\omega n(\mathrm{j}\omega)d/\mathrm{c}_0} \varGamma(\mathrm{j}\omega)^2\right| < \sqrt{2}\left|1 - \mathrm{e}^{-2\mathrm{j}\omega_0 n(\mathrm{j}\omega_0)d/\mathrm{c}_0} \varGamma(\mathrm{j}\omega_0)^2\right| \tag{13}$$

If there exist $\varDelta'$ such that condition (13) implies condition (11) for $\varDelta = \varDelta'$, then we have $B_{1/2} < B_{\varDelta'}$. For the zeroth order mode, condition (13) can be written

$$\left|n(\mathrm{j}\omega) + 2\frac{\sqrt{\mathrm{j}\omega\tau}\mathrm{c}_0}{\omega d}\right| \leq \sqrt{2}\frac{\omega_0}{\omega}\left|n(\mathrm{j}\omega_0) + 2\frac{\sqrt{\mathrm{j}\omega_0\tau}\mathrm{c}_0}{\omega_0 d}\right| \quad \Rightarrow \quad |n(\mathrm{j}\omega)| \leq \frac{\omega_0}{\omega}\left(\sqrt{2}\left|n(\mathrm{j}\omega_0) + 2\frac{\sqrt{\mathrm{j}\omega_0\tau}\mathrm{c}_0}{\omega_0 d}\right| + 2\frac{\sqrt{\omega\tau}\mathrm{c}_0}{\omega_0 d}\right) \tag{14}$$

Assuming narrow-band operation so that $\omega \approx \omega_0$, the last term can be written in terms of the skin depth at the resonance frequency $\delta = \sqrt{2/(\mu_0\sigma\omega_0)}$ as $\frac{\sqrt{\omega\tau}c_0}{\omega_0 d} \approx \frac{\delta}{\sqrt{2}d}$. This implies

$$|n(\mathrm{j}\omega)| \leq \frac{\omega_0}{\omega}\left(\sqrt{2}|n(\mathrm{j}\omega_0)| + \sqrt{2}(\sqrt{2}+1)\frac{\delta}{d}\right) \quad \Rightarrow \quad \Delta' = \sqrt{2}|n(\mathrm{j}\omega_0)| + (2+\sqrt{2})\frac{\delta}{d} \tag{15}$$

Finally, the inverse of the quality factor can be estimated as

$$\frac{1}{Q} = B_{1/2} \leq B_{\Delta'} \leq \frac{\Delta'/n_\infty}{1-\Delta'/(2n_\infty)} \approx \frac{3}{2}\frac{\Delta'}{n_\infty} = \frac{3\sqrt{2}}{2n_\infty}\left(|n(\mathrm{j}\omega_0)| + (\sqrt{2}+1)\frac{\delta}{d}\right) \tag{16}$$

This demonstrates that the quality factor of the resonator depends on whether the losses of the material or the metal walls are dominating: in the former case, there is no size dependence, in the latter case, the size dependence is similar to a vacuum-filled resonator.

4 Conclusions

Based on the physical bounds on metamaterials in [5], we have derived an estimate on the quality factor of the simplest size-independent metamaterial resonator proposed in [1]. The estimate shows that the quality factor can not be smaller than the smallest quality factor provided by the materials (proportional to $1/|n|$) or the vacuum-filled resonator with lossy walls (proportional to d/δ). In particular, the latter restriction implies a size dependence of the resonator when realistic constraints are taken into account.

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