Impulsive Electrostatic Waves Generated by a Point Charge in an unbounded Cold Magnetized Plasma

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Abstract

A problem of impulsive excitation of longitudinal waves emanated by a point source immersed in a cold, homogeneous and magnetized plasma is considered. The time evolution of the plasma response is inferred by use of Picard's method for solving the Volterra integral equation. This leads to an analytical representation of the resolvent kernel for the integral equation as a space pseudo differential operator. In order to find the solution, the convolution of the kernel with the source term is calculated. Such operation is readily performed with the aids of the Heaviside direct operational calculus. The method allows a simpler integral representation of the magnetized plasma response. This implicit formulation of the solution has two principal advantages. First, it may be used to devise the analytical expressions of the longitudinal waves as the temporal variable is sufficiently large. Such asymptotic representation provides us with some important insights to the physics of the upper hybrid plasma waves. Second, Neumann's series of Bessel functions for the solution may efficiently deduced. This is used to describe the dynamics of the waves especially during the transient phase of their evolution.

1. Introduction

In a recent paper [1], an analytical method to investigate the impulsive excitation of electrostatic waves by punctual antenna in a bounded magnetized plasma has been proposed. The primary intend was to provide a theoretical model for the experimental work published by Simonutti [2], on the upper hybrid wave excitation within a laboratory plasma.

In this paper, I aim to generalize the method to the problem of impulsive excitation within a cold non finite in extend magnetized plasma. In Ref. [1], a technique based on Picard's successive approximation solution for the time integral equation has been performed. As a result, the dynamics of the transient waveforms are analytically described within the Bessel-Fourier space, the finite extension of the plasma imposing all the possible vector-wave components. We note that, by use of the Heaviside direct operational calculus, the expression of the wave forms may be expressed, in the real space configuration, in terms of differential operators. This naturally stems when replacing the Fourier k-vector by the imaginary unit number multiplied by the spatial gradient operator. The resolvent kernel of the initial Volterra integral equation describing the wave propagation transforms this way to a pseudo differential operator. The main analytical procedures, based on the direct operational method, for the space inversion are described. The results consist then in explicit algebraic representations of the longitudinal plasma waves excited at the upper hybrid branches.

2. Basic Volterra Integral Equation and earlier Results

We wish to study the time evolution of electrostatic waves emanated by an antenna immersed within a cold anisotropic plasma. The antenna is considered as a point source concentrated after the fashion of Dirac delta function and localized at the origin O of the system of co-ordinates. The ion motion may be neglected if we aim to investigate high frequency perturbation of the material medium. Using a fluid model for the description of the evolution of electrons, the dynamics of the electric field potential $\phi(\vec{\mathrm{r}},t)$ for curl-free waves are governed by

$$\left(\partial_t^2+\omega_{uh}^2\right)\partial_t^2\vec{\nabla}^2\phi+\omega_c^2\omega_p^2\partial_z^2\phi=\left(\partial_t^2+\omega_c^2\right)\partial_t^2\vec{\nabla}^2\phi_{\mathrm{ext}}, \tag{1}$$

where $\omega_{uh}^2=\omega_c^2+\omega_p^2$, ω_c and ω_p are the cyclotron and plasma electron frequencies, respectively. In (1), the quantity $\phi_{\mathrm{ext}}(\vec{\mathrm{r}},t)$ designates the forcing source. Recourse to Heaviside's direct operational calculus provides us with an equivalent integral equation. Indeed, if we apply the operator defined by $(\partial_t^4+\omega_{uh}^2\partial_t^2)^{-1}$ throughout the dynamics equation, we find that

$$\phi(t) - \lambda \int_0^t h(t,\tau)\phi(\tau)d\tau = \Phi(t), \tag{2}$$

where

$$h(t,\tau) = \sin[\omega_{uh}(t-\tau)] - \omega_{uh}(t-\tau),$$

the prefactor λ represents the differential operator $\lambda = (\omega_c^2\omega_p^2/\omega_{uh}^3)(\partial_z^2/\vec{\nabla}^2)$. Note that, if the external excitation potential impresses an impulsive and extremely short perturbation in time, the source term of the Volterra integral equation of second kind (2) reads $\Phi(t) = \phi_C[\delta(t) - (\omega_p^2/\omega_{uh})H(t)\sin(\omega_{uh}t)]$, $\phi_C = (q/4\pi\varepsilon_0 r)$ is the free space Coulomb potential created by the antenna charge q at the position r, $H(t)$ and $\delta(t)$ designate the Heaviside unit step function and the Dirac delta function, respectively. This source term takes into account the physically acceptable initial value conditions.

By means of Picard's method, (2) has an unique solution given by

$$\phi(t) = \Phi(t) + \int_0^t Q(t,\tau)\Phi(\tau)d\tau,$$

where the resolvent kernel is obtained by summing the series of successive solutions which may be obtained by induction [1], [3]. Here, we find

$$Q(t,\tau) = -\omega_{uh}\chi^2 \sum_{m=0}^{\infty}(-1)^m \frac{[\omega_{uh}(t-\tau)]^{2m+3}}{(2m+3)!}\chi^m C_m^{(1)}[1/(2\chi)]. \tag{3}$$

In (3), $C_m^{(1)}(x)$ stands for the Gegenbauer or ultraspherical polynomial [4] of degree m and order 1 of the argument x and $\chi = (\omega_c\omega_p/\omega_{uh}^2)(\partial_z^2/\vec{\nabla}^2)^{1/2}$ represents a space pseudo differential operator. These results have been adapted from those of Ref. [1].

The integration with respect to time may be readily performed. Indeed, the time integral calculation reduces into convolution products of the kernel (3) with the source term $\Phi(t)$. Concerning the integration with respect to the space variable, the expression of the resolvent kernel indicates that each successive solution may be obtained by applying the space pseudo differential operator (χ^{2m}) to the function $1/(4\pi r) = 1/[4\pi(\rho^2+z^2)^{1/2}]$ of the position variables (here, $\vec{\mathrm{r}} = \rho\vec{\mathrm{e}}_\rho + z\vec{\mathrm{e}}_z$).

3. Space Inversion

A comprehensive review of the technique used on calculating various Heaviside operations has been proposed by Lindell [5]. This method may be efficiently applied to construct explicit solution for the problem.

First, one shows that

$$\frac{1}{(\vec{\nabla}^2)^m}\delta(\vec{\mathrm{r}}) = -\frac{1}{4\pi}\frac{r^{2m-3}}{[2(m-1)]!}. \tag{4}$$

Then, higher order derivatives with respect to the axial variable z have to be determined. We note the following formula, provided n a positive integer

$$\partial_z^{2n} r^{2n-1} = \frac{2^{2n}}{\pi}\Gamma^2(n+\tfrac{1}{2})\frac{1}{r}\left(\frac{\rho}{r}\right)^{2n}, \tag{5}$$

here, the symbol $\Gamma(x)$ refers to the Euler gamma function [4]. We note that both the identities (4) and (5) may be demonstrated by induction. Collecting all the results, the expression of the solution in the real space configuration may be formulated. This yields

$$\phi(\vec{\mathrm{r}},t) = \Phi(\vec{\mathrm{r}},t) + \phi_{\mathrm{ind}}(\vec{\mathrm{r}},t), \tag{6}$$

a summation of the source term of the Volterra integral equation $\Phi(\vec{\mathrm{r}},t)$ and $\phi_{\mathrm{ind}}(\vec{\mathrm{r}},t)$ which represents the electric potential induced by the magnetized plasma. One has

$$\phi_{\mathrm{ind}}(\vec{\mathrm{r}},t) = \frac{\mathrm{q}}{4\pi\varepsilon_0 r}H(t)\left[\omega_{uh}S_S(r,t) - (\omega_p^2/\omega_{uh})S_L(r,t)\right],$$

where

$$S_S(r,t) = \sum_{k=0}^{+\infty} (-1)^{k+1} I_k(r)(\omega_{uh}t)^{2k+3} /(2k+3)!, \tag{7}$$

$$S_L(r,t) = \sum_{k=0}^{+\infty} (-1)^{k+1} I_k(r) U_{2k+5}(2\omega_{uh}t, 0), \tag{8}$$

with

$$I_k(r) = \frac{2}{\pi} \int_0^{\pi/2} (\beta\rho / r)^{k+2} \cos^{k+2}\theta \; C_k^{(1)}[r/(2\beta\rho\cos\theta)]\, d\theta, \tag{9}$$

and the notations: $U_k(x,y)$ for the two variables Lommel functions of second kind and of order k [6] and $\beta = (\omega_c \omega_p / \omega_{uh}^2)$. The plasma response is then put as a series. Because the element of each series exhibits integral, the computation of these series as they stand turns out however to be very arduous. Recourse to further arrangement and simplification happens to be necessary. We take advantage of the fact that both Gegenbauer's polynomials and the two variables Lommel functions may be represented under various forms to derive some interesting analytical results.

4. Asymptotic Representation

Choosing some suitable forms for $1/[(2k+3)!]$ and the Lommel function, the expressions (7) and (8) may be transformed into integrals, the summation of the series becoming analytically tractable. Using the classical technique of stationary phase for asymptotic expansion of integrals [6], one can find the representation of the solution at very elapsed time from the switch-on. If we limit ourselves to the leading terms of the plasma response, the plasma response reads

$$\phi_{ind}(\vec{\mathrm{r}}, \mathrm{t}) \approx -\frac{\mathrm{q}}{4\pi\varepsilon_0 r} \frac{\omega_{uh}}{(\omega_{uh}t)^{1/2}} [A_- \cos(\alpha_- \omega_{uh} t - \pi/4) + A_+ \cos(\alpha_+ \omega_{uh} t + \pi/4)], \tag{10}$$

where, $\alpha_\pm = [(1/2)(1 \pm (1-b^2)^{1/2}]^{1/2}$, $b = 2\beta(\rho/r)$ and

$$A_\pm = \mp \frac{b}{\sqrt{2\pi}} \frac{1}{(1-b^2)^{1/4}} \frac{1}{\alpha_\pm^{1/2}} .$$

Both the upper branch for the resonance of the Z-mode (phase with α_+) and the lower branch for the whistler mode(phase with α_-) contribute to the total field.[7] Globally the magnitude of the waves decreases as $(\omega_{uh}t)^{-1/2}$. The spatial dependence of the wave amplitude is confined at its variation with respect to the b parameter. We find that the phase terms do not contain the common wave-vector component $\vec{\mathrm{k}} \cdot \vec{\mathrm{r}}$. This clearly indicates that the group speed of each mode, oriented with the observation direction $\vec{\mathrm{r}}$, turns out to be perpendicular to the phase speed vector. Other physical propagation characteristics of these waves may be deduced from the investigation.

4. Neumann Series

We also note that the temporal dependence of the integrand exhibits a sine function. One has

$$\sin(\omega_{uh} t y) = 2\sum_{n=0}^{+\infty} (-1)^n J_{2n+1}(\omega_{uh}t) T_{2n+1}(y), \tag{11}$$

where J_n is the Bessel function and T_n the Chebyshev polynomial. Plugging the Bessel series into the solution integral, this solution is reduced to a Neumann series with respect to the time, and the integration is relegated in the coefficients of the series. It may be shown that each coefficient take the general form of elliptic integrals, namely

$$f(u) = \int \frac{P(u)}{\sqrt{u(1-u)(\alpha_-^2 - u)(\alpha_+^2 - u)}} du, \tag{12}$$

where $P(u)$ stands for a polynomial. In general, the integral (17) may be carried out analytically and details of the different procedures for the calculation are proposed in Standard Tables of integrals (e.g., Ref. [8] p. 80). The results consist then in combinations of the Legendre elliptic integrals [9]. The first four coefficients of each series are determined analytically from existing mathematical tables of integrals and then a recurrence formula pertain us to calculate all coefficients for $n > 5$. Note that series converges rapidly with the cylindrical Bessel functions and the

coefficients determined for $n = 5$ corresponds to $J_{11}(\omega_{uh} t)$. Such representation turns out to be very useful for numerical computations.

5. Conclusion

Simonutti integral expression for the impulsive response of a cold magnetized and unbounded plasma, subject to external point charge, is a series of convolution products.[2] Analytical and numerical computations of the solution may, in some physical situations, be arduous. In the paper a new method of the solution is proposed. Based on the integral equation solving and the direct operational method, the formalism provides interesting expressions of the solution. Applying the technique to handle the continuous excitation in order to generate the resonance cone phenomenon at the upper hybrid consists in a first generalization of this work.

6. References

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