Critical points in the applications of geometrical optics to numerical simulation of lightning-related spectrograms

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Abstract

Analysis of spectrograms observed on satellites is one of the main instruments for studying wave phenomena in the Earth's plasmasphere. Special features of VLF spectrograms related to lightning strokes include the facts that the location of emission source is often known accurately, and that the whistler mode of propagation is beyond question. These facts make it possible to numerically simulate the corresponding spectrograms, using geometrical optics as the main tool, and this is becoming a new line in whistler studies. However, most of the work in this direction is based on intuitive ideas, rather than on well founded concepts. This limits the numerically simulated spectrograms to being no more than interesting and useful illustrations of real ones. To make numerical simulation a powerful and reliable tool in ionospheric and magnetospheric studies, the approach to numerical modeling should be well substantiated, and the range of validity of the method should be clarified. The points that need to be explicated include:

- the wave field expansion into geometrical optics wave packets in an inhomogeneous medium;
- construction of frequency-time plots on spectrograms, i.e. finite-width curves on which the spectral intensity differs from zero;
- determining the time-dependent spectral amplitude as a function of frequency and time from the wave packet amplitude, with the account of the evolution of the latter in space and time.

Some of these and related questions are discussed in this paper. Special attention is paid to the properties and the presentation of the initial field, to the wave packet spreading in the course of propagation, and to the choice of the width of the frequency bin on a spectrogram in relation to the characteristic time of spectrum evolution. The intrinsic limitations of the method are discussed in detail.

1. Introduction

The generally accepted phenomenology of formation the spectrograms related to lightning-induced emissions is as follows. Lightning stroke creates electromagnetic radiation. While propagating in the Earth-ionosphere waveguide, the emission partly leaks into the ionosphere, the initial **k**-vector of the emission being vertical at the top of the F region. An extended region around the lightning stroke illuminated by lightning emission serves as a secondary source that illuminates the magnetosphere. These emissions propagate in the magnetosphere in nonducted mode being subject to lower hybrid resonance (LHR) reflection.

An intuitive method for numerical modeling of these spectrograms consists in calculating the group delay time for 2D array of trajectories in the frame of geometrical optics, starting all trajectories (characterized by frequency f and initial latitude λ) from one same altitude with the vertical direction of wave normal vectors. This procedure, which gives the frequency-time plot on a spectrogram, is supplemented by evaluation of wave packet's amplitudes and relating them to spectral intensity.

The questions which arise under more careful consideration of the problem are the following. As the initial field is not a subject of geometrical optics (GO), then how to expand it into GO wave packets in an inhomogeneous medium? What does the initial wave energy distribution over frequency and latitude really mean, and how to use it? If the field is not a quasi-plane wave, what does the vertical direction of the initial wave normal vector assume? These and related questions are discussed in the present paper.

2. Expansion of the wave field into discrete wave packets

To apply methods of geometrical optics in the problem under discussion, a method should be developed for expanding a wave field into GO wave packets. Below, we suggest a way for doing this. We proceed from the case of a homogeneous medium, and then extend the method to an inhomogeneous one.

2.1. Homogeneous case

Expansion of the function $f(x)$ in Fourier integral is given by

$$f(x) = \int_{-\infty}^{\infty} f(k) e^{ikx} \frac{dk}{2\pi} \; ; \quad f(k) = \int_{-\infty}^{\infty} f(x') e^{-ikx'} dx' \; . \tag{1}$$

Dividing the axis k into intervals of the length $2\delta k$ centered on k_n, and writing the integral over k as the sum of integrals over these intervals we get

$$f(x) = \sum_n \int_{k_n-\delta k}^{k_n+\delta k} f(k) e^{ikx} \frac{dk}{2\pi} \; . \tag{2}$$

Substituting $f(k)$ determined by (1) into (2) and changing the order of integrals we obtain, after integration over dk

$$f(x) = \sum_n \int_{-\infty}^{\infty} f(x') e^{ik_n(x-x')} \frac{\sin[\delta k(x-x')]}{\pi(x-x')} dx' \; . \tag{3}$$

The relation (3) has the form of expansion

$$f(x) = \sum_n F_n(x) e^{ik_n x} \; ; \quad F_n(x) = \int_{-\infty}^{\infty} f(x') e^{-ik_n x'} \sigma(x-x') dx' \; , \tag{4}$$

where

$$\sigma(z) \equiv \int_{-\delta k}^{\delta k} e^{ikz} \frac{dk}{2\pi} = \frac{\sin(\delta k \cdot z)}{\pi z} \tag{5}$$

2.2. Field expansion in an inhomogeneous medium

In an inhomogeneous medium, a dispersion relation reads

$$\omega = H(k, x) \; , \tag{6}$$

while the desired expansion of the wave field has the form

$$\sum_n \Phi_n(x,t) e^{i\Psi_n(x,t)} \; , \tag{7}$$

where each phase $\Psi_n(x,t)$ satisfies eikonal equation

$$-\frac{\partial \Psi_n}{\partial t} = H\left(\frac{\partial \Psi_n}{\partial x}, x\right) \; . \tag{8}$$

With α being the value of $k = \partial\Psi/\partial x$ at $x = 0,\ t = 0$, and $\omega(\alpha) = H(\alpha, 0)$, the full integral of equation (8) takes the form

$$\Psi(x,t,\alpha) = \int_0^x k(x',\alpha)\, dx' - \omega(\alpha)\, t + \text{const.} \tag{9}$$

We denote the spatial part of eikonal (9) by $\psi(x,\alpha)$:

$$\psi(x,\alpha) = \int_0^x k(x',\alpha) dx' \; , \tag{10}$$

and look for the expansion of the initial field $f(x)$ in the form:

$$f(x) = \int_{-\infty}^{\infty} C(\alpha) e^{i\psi(x,\,\alpha)} \frac{d\alpha}{2\pi} \; . \tag{11}$$

Assuming that the main contribution to the integral in (11) comes from a small enough region of α close to some α_0 we may write

$$f(x) = \int_{-\infty}^{\infty} C(\alpha) \, \exp\left[i\psi(x,\alpha_0) + i\frac{\partial \psi(x,\alpha_0)}{\partial \alpha}(\alpha-\alpha_0)\right] \frac{d\alpha}{2\pi} \; . \tag{12}$$

With the notation

$$\psi(x,\alpha_0) = \psi_0(x) \; ; \qquad \frac{\partial \psi(x,\alpha_0)}{\partial \alpha} = s(x) \; ; \qquad (\alpha-\alpha_0) = \lambda \; , \tag{13}$$

we get from (12)

$$f(s)e^{-i\psi_0(s)} = \int_{-\infty}^{\infty} C(\lambda)\; e^{i\lambda s}\frac{d\lambda}{2\pi} \; ; \quad C(\lambda) = \int_{-\infty}^{\infty} f(s') \, \exp[-i\psi_0(s') - i\lambda s'] ds' \; . \tag{14}$$

Similar to homogeneous case, discrete expansion in variables $s,\ \lambda_n$ may be written as

$$\begin{aligned} f(s) \quad &= \quad \sum_n F_n(s) \, \exp[i\psi_0(s) + i\lambda_n \, s] \\ F_n(s) \quad &= \quad \int_{-\infty}^{\infty} f(s') \, \exp[-i\psi_0(s) - i\lambda_n s'] \frac{\sin[\delta\lambda(s-s')]}{\pi(s-s')} ds' \end{aligned} \tag{15}$$

Then (returning from s to original variable x) the expansion of the field $f(x,t)$ takes the required form:

$$\begin{aligned} f(x,t) \quad &= \quad \sum_n \Phi_n(x,t) \, \exp[i\Psi_n(x,t)] \; ; \\ \Phi_n(x,0) \quad &= \quad \int_{-\infty}^{\infty} f(x',0) \, \exp[-i\Psi_n(x',0)] \, \frac{\sin\{\delta\lambda[s(x)-s(x')]\}}{\pi[s(x)-s(x')]} \frac{ds(x')}{dx'} dx' \; , \end{aligned} \tag{16}$$

where

$$\Psi_n(z,t) = \psi(z,\alpha_n) - \omega(\alpha_n)\; t \equiv \int_0^z k(x',\alpha_n) \; dx' - \omega(\alpha_n)\; t \; . \tag{17}$$

3. Initial field and its basic properties

Distribution of energy emitted by the lightning stroke over frequency and latitude is assumed to be known, but what does it really mean? All wave vectors are nearly vertical at the beginning. What does this assertion mean in the case when the field cannot be characterized by a wave vector at each point? To answer these questions, we turn to the wave field decomposition into wave packets:

$$f(\mathbf{r},t) = \sum_{\mathbf{n}} \Phi_{\mathbf{n}}(\mathbf{r},t) e^{i\Psi_{\mathbf{n}}(\mathbf{r},\,t)} \; , \tag{18}$$

where

$$\Psi_{\mathbf{n}}(\mathbf{r},t) = \psi(\mathbf{r},\boldsymbol{\alpha}_{\mathbf{n}}) - \omega(\boldsymbol{\alpha}_{\mathbf{n}}) \; t \; . \tag{19}$$

In 2D case under discussion, $\boldsymbol{\alpha}_{\mathbf{n}} = (\alpha_{1n}, \alpha_{2m})$. For a given $\boldsymbol{\alpha}_{\mathbf{n}}$,

$$k_r = k_r(\alpha_{1n},\alpha_{2m},\lambda), \quad k_\phi = k_\phi(\alpha_{1n},\alpha_{2m},\lambda).$$

Equations

$$k_\phi(\alpha_{1n},\alpha_{2m},\lambda_{nm}) = 0; \quad \omega_{nm} = \omega(\alpha_{1n},\; \alpha_{2m}) \; , \tag{20}$$

express α_{1n} and α_{2m} through ω_{nm} and λ_{nm}. Using them as independent parameters of the wave packets, ω_ν and λ_μ, we get the expansion of the initial field in the form

$$f(\mathbf{r}, t=0) = \sum_{\mu,\nu} \Phi_{\mu,\nu}(\mathbf{r}, t=0) e^{i\psi(\mathbf{r};\, \omega_\nu, \lambda_\mu)} \,. \tag{21}$$

With such a presentation, the notion of initial frequency-latitude distribution of the lightning-emitted wave energy gets clear physical sense.

4. Constructing the frequency-time plot using Storey's definition and physical explanation of a group front

How to find the time at which the wave packet "comes" at the satellite using the notion of group front? The general solution of the equations of GO

$$\mathbf{r} = \mathbf{r}(\mathbf{r}_0,\ \mathbf{k}_0,\ t) \ , \quad \mathbf{k} = \mathbf{k}(\mathbf{r}_0,\ \mathbf{k}_0,\ t) \quad \text{while} \quad \omega(\mathbf{k},\ \mathbf{r}) = \omega(\mathbf{k}_0,\ \mathbf{r}_0) \,. \tag{22}$$

Since all rays start from a single altitude with the vertical wave normal, there are only two independent initial variables: the wave frequency ω and the initial latitude λ_0. Solution to the equations of GO optics for L, λ, and $\mathbf{k}$ in general form is

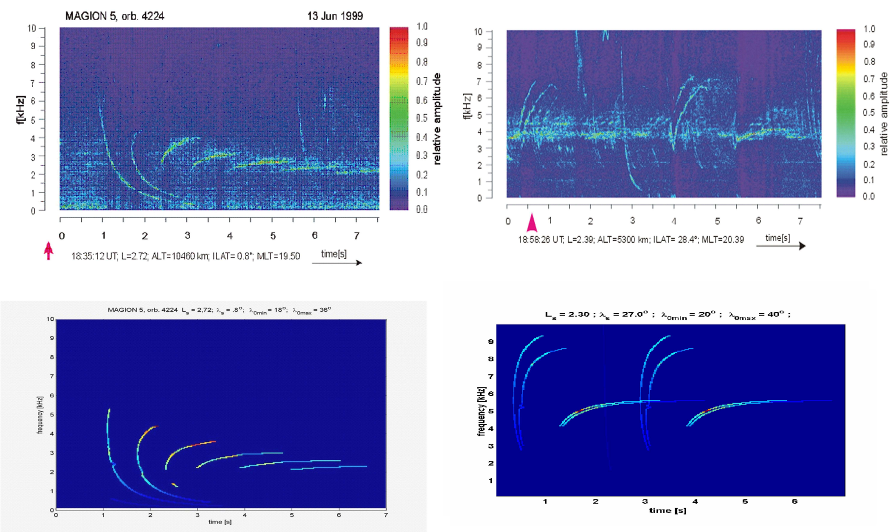
$$\begin{gathered} L = L(\lambda_0,\ \omega,\ t) \ ; \lambda = \lambda(\lambda_0,\ \omega,\ t) \ ; \\ \mathbf{k} = \mathbf{k}(\lambda_0,\ \omega,\ t) \ ; \end{gathered} \tag{23}$$

The first two relations in (23) define in a parametric way the time t and the initial latitude λ_0 as functions of ω, λ, and L:

$$t = t(\omega;\ \lambda,\ L) \ ; \quad \lambda_0 = (\omega;\ \lambda,\ L) \,. \tag{24}$$

The first function in (24) defines the time when the group front for the frequency ω crosses the satellite position λ, L, thus it yields the time-frequency curves on the spectrogram. The second function determines the initial latitude for the frequency ω on each branch.

Finding the time-frequency curves is the first step in simulating spectrograms. The second one is determination of spectral intensity from wave packet's amplitudes. This problem is considered in detail in [1], where fundamental limitations of the method are also discussed. Examples of real and simulated spectrograms are shown below.



6. Reference

1. D. R. Shklyar, "Key problems in numerical simulation of spectrograms related to emissions caused by lightning strokes," *Geomagnetism and Aeronomy,* **45,** No 4, 2005, pp. 474-487.