Pulse Shapes for OFDM with Maximum Energy Concentration in the Spectral Main-lobe

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***Abstract*—Various characteristics of the OFDM communication system are affected by the spectrum of the shaping pulse. For reasons that include bandwidth efficiency and resilience to interference, it is desirable to have a nearly flat spectral main-lobe and attenuated side-lobes that decay rapidly. In order to realize such a pulse, we formulate an optimization problem for maximizing the energy concentration in the spectral main lobe while satisfying certain smoothness constraints in the time-domain. The resulting optimization problem is a non-convex fractional programming problem (FPP). We show in this paper that for the pulse shaping problem, the FPP is solved globally, by iteratively solving a family of concave quadratic programs with linear constraints. The overall method is efficient, since each quadratic program is solved in closed form. Numerical results show that the optimized pulses are spectrally efficient, as well as robust to channel imperfections.**

I. Introduction

Since its adoption by the European Digital Audio Broadcasting (DAB) standard in 1987, orthogonal frequency-division multiplexing (OFDM) has gained widespread popularity as a wireless transmission technology. By transmitting digitally encoded symbols in parallel over several orthogonal sub-carriers, and by introducing the cyclic-prefix, OFDM reduces a slow-fading frequency-selective channel into several frequency flat channels. Thus, the task of equalization at the receiver is greatly simplified, which makes the system robust to inter-symbol interference (ISI). The inherent suitability of OFDM to wireless channels has led to it being adopted by a number of standards including the IEEE 802.11a, 802.11g and HIPERLAN/2.

In a channel prone to inter-carrier interference (ICI), the shape of the pulse spectrum influences the received signal to noise and interference ratio (SINR), which in turn determines the system's bit error rate (BER) performance. While a nearly flat main-lobe results in a high average signal power, the magnitude of the interference is reduced as the side-lobes are attenuated. Thus, a pulse that has very little of its energy outside the main-lobe is expected to be perform better in an ICI channel. In [1] for instance, pulse shapes have been proposed that satisfy Nyquist's first criteria for zero interference in the frequency domain. These pulses have side lobes with small amplitudes, and are therefore robust to ICI.

The asymptotic decay rate (ADR) of the tail of the signal spectrum depends on the smoothness of the pulse shaping function in the time domain. Denoting the ADR by the integer parameter η, it is a well known result [2] that pulses that are continuous on the entire real line have $\eta \geq 2$, while those that are also continuously differentiable have $\eta \geq 3$. In addition, the smoothness properties of the pulse are important from a hardware implementation point of view.

In order to realize a pulse with the above properties, we formulate a constrained optimization problem that involves maximizing the ratio of two convex functions: energy contained in the main-lobe of the pulse spectrum, and its net energy. Problems of this kind are known as convex-convex type fractional programming problems (FPP) [3]. FPPs constitute a class of global optimization problems since the objective function possesses multiple local extrema.

Dinkelbach's parametric method [4] is one of the earliest techniques devised for solving a general FPP. We show in this paper and [5] that this method is particularly suited for our pulse design problem, since it reduces to the iterative solution of a family of concave quadratic programs with linear constraints. The overall algorithm is efficient, since each sub-problem is solved in closed form.

We present design examples using the proposed method, and study the performance of the pulses so obtained under non-ideal channel conditions. Our experiments show that besides being optimal in terms of main-lobe energy concentration, the optimized pulses are more resilient to channel imperfections compared to other well-known pulses.

II. System Model

We consider a K-carrier OFDM system, with *symbol* duration after serial-to-parallel conversion T. In order to achieve better spectral characteristics of the transmitted signal, a symbol is transmitted over the duration T_p which exceeds the symbol duration as $T_p = T(1+\alpha)$, where $\alpha \in [0,\, 1]$ is the normalized excess time duration parameter, commonly known as the roll-off factor. We denote the interval $[-T_p/2,\, T_p/2]$ as the *block* interval. Thus, the complex baseband transmit signal $s(t)$ over the duration of a block is

$$s(t) = \sum_{k=0}^{K-1} x_k p(t) e^{j2\pi f_k t}, \tag{1}$$

where x_k is the complex valued information symbol transmitted on the k^{th} sub-carrier f_k, and $p(t)$ is the pulse shaping function, time limited to the block interval.

The sub-carrier frequency separation is related to the symbol rate $1/T$ as $f_k - f_m = \frac{k-m}{T}$, and mutual orthogonality

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of the sub-carriers is ensured by requiring that for each $l = \pm 1, \pm 2, \ldots$, $p(t)$ has spectral nulls at the frequencies l/T:

$$\int_{-T_p/2}^{T_p/2} p(t) e^{-j2\pi(f_k - f_l)t} \, \mathrm{d}t = 0, \qquad \text{when } l \neq k. \qquad (2)$$

A. Inter Carrier Interference (ICI) model

The carrier modulated passband signal corrupted by additive noise at the front end of the receiver is

$$\tilde{s}(t) = \Re\{\sqrt{2} s(t) e^{j2\pi f_c t}\} + w(t),$$

where f_c is the carrier frequency, and $w(t)$ is the white Gaussian noise process with two-sided power spectral density (PSD) $N_0/2$.

Non-idealities due to channel distortion, or loss of time or frequency synchronization can be incorporated by introducing a frequency-offset term δf in the receiver's local oscillator (LO) signal $e^{j2\pi(-f_c+\delta f)t}$. We model δf as a random variable that takes values in the interval $[-\Delta f, \Delta f]$ according to the density function $\zeta_{\delta f}$. Carrier demodulation of $\tilde{s}$ with the imperfect LO signal followed by correlation with $e^{-j2\pi f_k t}$ yields the soft estimate r_k of the k^{th} transmitted symbol as

$$r_k = x_k P(-\delta f) + \sum_{\substack{l=0 \\ l \neq k}}^{K-1} x_l P\big(f_k - f_l - \delta f\big) + w_k. \qquad (3)$$

The first term in (3) is the desired signal component, while the second and the last terms are the noise components due to ICI and Gaussian noise respectively. The variance of the Gaussian random variable w_k is $\mathrm{E}[|w_k|^2] = N_0 T_p$. It is assumed that the random variables x_k and $P(-\delta f)$ are statistically independent. Assuming further that the symbols x_k are random variables with mean zero, variance E_s, and are mutually uncorrelated: $E[x_k^* x_l] = E_s \delta_k(l)$; the signal power σ_k^2, and net ICI power ρ_k^2 experienced by the k^{th} sub-carrier is easily derived from (3) as

$$\sigma_k^2 = E_s \, \mathrm{E}\big[|P(-\delta f)|^2\big], \; \rho_k^2 = E_s \sum_{\substack{l=0 \\ l \neq k}}^{K-1} \mathrm{E}\big[|P(f_k - f_l - \delta f)|^2\big]. \qquad (4)$$

Since the support Δf of the density function $\zeta_{\delta f}$ is small compared with the inter-carrier separation $1/T$, we infer from (4) that in order for the received signal energy to be high, $P(f)$ should be nearly flat in an interval around $f = 0$. For the same reason, the dependence of the ICI power on $P(f)$ in small intervals around $f = l/T$ for $l \neq 0$ in (4) suggests that a reduction in ICI can be obtained if the side-lobes of P are constrained to small magnitudes. The above observations motivate the formulation of the optimization problem in Section IV as the maximization of the main-lobe energy to net energy ratio of $p(t)$.

III. NYQUIST-I PULSES

Given the normalized excess time duration parameter α, we consider functions $g(t)$ that are continuous in the open interval $(-\alpha T/2, \alpha T/2)$. The pulse waveform $p(\alpha; t) = p(\alpha; g)(t)$, with support in the interval $[\frac{-T(1+\alpha)}{2}, \frac{T(1+\alpha)}{2}]$, is constructed from $g(t)$ as

$$\begin{aligned} p(\alpha; g)(t) &= \frac{1}{T} \operatorname{rect}\big(t/(1-\alpha)T\big) \\ &+ \frac{1}{2T}\Big[\operatorname{rect}\big((t - T/2)/\alpha T\big) + \operatorname{rect}\big((t + T/2)/\alpha T\big)\Big] \\ &+ \frac{1}{2T}\Big[g(t - T/2) + g(-t - T/2)\Big]. \end{aligned} \qquad (5)$$

We note that the above form imposes an even symmetry on $p(\alpha; g)$, with the result that the Fourier transform $P(\alpha; G)$ of $p(\alpha; g)$ is purely real and even.

In view of Nyquist's first criteria for zero interference, $p(\alpha; g)$ will satisfy the the orthogonality conditions (2), if it has odd symmetry about the point $(t, p(t)) = (\frac{T}{2}, \frac{1}{2T})$ in the interval $[\frac{T(1-\alpha)}{2}, \frac{T(1+\alpha)}{2}]$. Accordingly, $g(t)$ is constrained to be *odd* symmetric.

The Fourier transform $G(f)$ of $g(t)$ is purely imaginary and odd. By defining the function $\tilde{G}$ as $\tilde{G}(f) = \Im\{G(f)\}$, where $\Im\{\cdot\}$ denotes the imaginary part, the Fourier transform of $p(\alpha; g)$ is obtained from (5) as

$$P(\alpha; \tilde{G})(f) = \operatorname{sinc}(fT) \cos(\pi f T \alpha) + \frac{1}{T} \tilde{G}(f) \sin(\pi f T).$$

Hereafter, without loss of generality, we normalize T to unity. Consequently, the main lobe of $P(f)$ is supported on the frequency interval $[-1, 1]$.

In order to formulate our signal design problem in a finite dimensional setting, we represent g in an orthogonal set of functions $\{\phi_n\}_{n=1}^{\infty}$ that constitutes a basis for the linear space of square integrable, odd functions on the interval $[-1, 1]$. Thus,

$$g(t) = \sum_{n=1}^{N} b_n \phi_n(2t/\alpha), \quad t \in [-\alpha/2, \alpha/2], \qquad (6)$$

where $b_n \in \mathbb{R}$ are the unknown coefficients.

A. Main-lobe energy and net energy of $p(t)$

As a consequence of the normalization $T = 1$, the frequency domain main-lobe is confined to the interval $[-1, 1]$. The following expression for the main-lobe energy of $p(\alpha; g)$ in terms of the unknown coefficient vector $\mathbf{b} \in \mathbb{R}^N$ is derived in [5]:

$$\|P(\alpha; \mathbf{b})\|_1^2 = \mathbf{b}^T A(\alpha) \mathbf{b} + 2\mathbf{q}^T(\alpha) \mathbf{b} + \kappa_f(\alpha).$$

The symmetric and positive-definite matrix $A(\alpha) \in \mathbb{R}^{N \times N}$, and the vector $\mathbf{q}(\alpha) \in \mathbb{R}^N$ depend on the basis functions, while the term $\kappa_f(\alpha)$ depends only on α. Since $A(\alpha)$ is positive definite, $\|P(\alpha; \mathbf{b})\|_1^2$ is a convex function of $\mathbf{b}$.

From (5), the net energy of $p(\alpha; g)$ is

$$\|P(\alpha; \mathbf{b})\|_\infty^2 = \|p(\alpha; g)\|_{(1+\alpha)/2}^2 = \frac{1}{2} \|g\|_{\alpha/2}^2 + \kappa_t(\alpha), \qquad (7)$$

where $\kappa_t(\alpha) = 1 - \alpha/2$ is the energy due to the rectangular components.

From the orthogonality of ϕ_n, and by defining the diagonal matrix $M(\alpha) \in \mathbb{R}^{N\times N}$ as

$$M(\alpha) := \frac{\alpha}{4}\,\mathrm{diag}(\|\phi_n\|_1^2, \ldots, \|\phi_N\|_1^2), \tag{8}$$

$\|P(\alpha; \mathbf{b})\|_\infty^2$ is obtained in matrix notation as

$$\|P(\alpha; \mathbf{b})\|_\infty^2 = \mathbf{b}^T M(\alpha)\mathbf{b} + \kappa_t(\alpha). \tag{9}$$

It is clear that the diagonal matrix M is positive definite, which implies that $\|P(\alpha; \mathbf{b})\|_\infty^2$ is a strictly convex function of $\mathbf{b}$.

B. *Smoothness constraints*

For the pulse spectrum to decay asymptotically as f^{-2} ($\eta = 2$), $p(\alpha; g)$ is required to be continuous on the entire real line. An ADR $\eta = 3$ requires that $p(\alpha; g)$ is also continuously differentiable everywhere. Thus, the boundary condition $g(\alpha T/2) = -1$ ensures an ADR $\eta \geq 2$, while the additional boundary condition $\frac{\mathrm{d}}{\mathrm{d}t}\, g(t)\big|_{t\uparrow\alpha T/2} = 0$ results in an ADR $\eta \geq 3$ [2].

By defining the matrix $C \in \mathbb{R}^{N\times\eta-1}$ as $C_{(n,l)} = \phi_n^{(l-1)}(1)$, where $f^{(l)}$ denotes the l^{th} derivative of f, and $\mathbf{d} \in \mathbb{R}^{\eta-1}$ as

$$\mathbf{d}_l := \begin{cases} -1 & l = 1, \\ 0 & 2 \leq l \leq \eta - 1, \end{cases}$$

the above smoothness constraints are written as: $C^T\mathbf{b} = \mathbf{d}$.

IV. Problem formulation and solution methodology

Having obtained expressions for the main-lobe and total energies of the Nyquist pulse $p(\alpha; t)$ in the previous section, we now formulate an optimization problem for maximizing the ratio of these two quantities.

Let $\xi(\mathbf{b})$ denote the main-lobe energy concentration of the pulse $p(\mathbf{b})$, *i.e.*

$$\xi(\mathbf{b}) := \frac{\|P(\mathbf{b})\|_1^2}{\|P(\mathbf{b})\|_\infty^2} = \frac{\mathbf{b}^T A\mathbf{b} + 2\mathbf{q}^T\mathbf{b} + \kappa_f}{\mathbf{b}^T M\mathbf{b} + \kappa_t}. \tag{10}$$

We consider the following constrained optimization problem

$$\text{(FP)} \qquad \begin{array}{cc} \displaystyle\max_{\mathbf{b}\in\mathbb{R}^N} & \xi(\mathbf{b}) \\ \text{s.t.} & C^T\mathbf{b} = \mathbf{d}, \end{array}$$

where η is chosen according to the ADR requirement.

We see from the positive definiteness of the matrices A and M that $\xi : \mathbb{R}^N \to \mathbb{R}$ is the ratio of two convex functions. Hence (FP) is a convex-convex type fractional-programming problem (FPP). Due to the lack of convexity, FPPs in general possesses multiple local maxima and are classified as global optimization problems. Consequently, the commonly used techniques of numerical optimization are not guaranteed to yield globally optimal solutions. For a comprehensive introduction to problems of this type and solution strategies, we refer the reader to the article by Benson [3].

A. *Solution methodology*

While the convex-convex FPP is difficult to solve in the general case, we will show in the following that a globally optimal solution to the fractional program (FP) can be obtained by iteratively solving a family of concave quadratic problems. Further, since each sub-problem is solved in closed form, the overall method is efficient.

Following Dinkelbach's parametric transformation, we introduce the parameter $\gamma \in \mathbb{R}$, and the function $F(\mathbf{b}; \gamma)$ associated with the constrained-maximization problem (FP) as

$$F(\mathbf{b};\, \gamma) := \|P(\mathbf{b})\|_1^2 - \gamma\, \|P(\mathbf{b})\|_\infty^2. \tag{11}$$

Let $\mathcal{B}$ denote the feasible set $\mathcal{B} := \{\mathbf{b} \in \mathbb{R}^n \,|\, C^T\mathbf{b} = \mathbf{d}\}$. The following two theorems are due to Dinkelbach [4].

Theorem 1: Define the function $\pi : \mathbb{R} \to \mathbb{R}$ as

$$\pi(\gamma) := \max_{\mathbf{b}\in B} F(\mathbf{b};\, \gamma), \tag{12}$$

and let $\mathbf{b}(\gamma)$ denote the optimal solution set: $\mathbf{b}(\gamma) := \arg\max_{\mathbf{b}\in\mathcal{B}} F(\mathbf{b};\, \gamma)$. If $\gamma^* \in (0, 1)$ is such that $\pi(\gamma^*) = 0$, then γ^* is the optimal value of (FP), and $\mathbf{b}^* := \mathbf{b}(\gamma^*)$ is an optimal solution.

Theorem 2: The function $\pi : \mathbb{R} \to \mathbb{R}$ is strictly decreasing.

By virtue of Theorem 1, the optimization problem (FP) is is equivalent to finding the root of the function $\pi(\gamma)$, which Theorem 2 shows to be rather well behaved. Clearly, the success of the method that theorems 1 and 2 suggest depends on how tightly we are able to bracket γ^* and how efficiently the family of optimization problems (12) is solved.

We introduce the matrix $Z \in \mathbb{R}^{N\times N}$ defined as $Z(\gamma) := A - \gamma M$, the function $\kappa(\gamma) := \kappa_f - \gamma\kappa_t$, and rewrite (11) as

$$\begin{aligned} F(\mathbf{b};\, \gamma) &= \mathbf{b}^T(A - \gamma M)\mathbf{b} + 2\mathbf{q}^T\mathbf{b} + \kappa_f - \gamma\kappa_t \\ &= \mathbf{b}^T Z(\gamma)\mathbf{b} + 2\mathbf{q}^T\mathbf{b} + \kappa(\gamma). \end{aligned} \tag{13}$$

Thus $\pi(\gamma)$ is the solution of the following quadratic program with linear constraints:

$$\text{(QP)} \qquad \begin{array}{cc} \displaystyle\max_{\mathbf{b}\in\mathbb{R}^n} & \mathbf{b}^T Z(\gamma)\mathbf{b} + 2\mathbf{q}^T\mathbf{b} + \kappa(\gamma) \\ \text{s.t.} & C^T\mathbf{b} = \mathbf{d}. \end{array}$$

Note that (QP) in general is non-concave, and is therefore a global optimization problem.

Let $\gamma^\dagger$ denote the largest eigenvalue of $M^{-1}A$. We show in [5] that $Z(\gamma)$ is negative definite for all $\gamma > \gamma^\dagger$. Thus (QP) is a concave quadratic program for γ in the range $\gamma^\dagger < \gamma < 1$. What is more important is that for our pulse shaping problem, the optimal value γ^* is always contained in the interval $\gamma^* \in (\gamma^\dagger, 1)$. Due to the negative definiteness of $Z(\gamma)$ in this interval, the method of Lagrange multipliers provides a closed form solution of (QP). In view of the strict decrease of $\pi(\gamma)$, a bisection algorithm applied to π in the interval $(\gamma^\dagger,\, 1)$ then locates the root of π in a few iterations.

V. Numerical Results

In this section, we study the optimal pulses obtained by solving the fractional program (FP) with various choices of the design parameters α and η. We discuss the properties of the resulting pulses with regard to their resilience to channel imperfections. The performance of the optimal pulses is compared to the raised-cosine pulse (*rc*) which has $\eta = 3$ and the "better than" raised-cosine pulse (*btrc*) [6] which has $\eta = 2$. We choose the basis set $\{\phi_n\}_{n=1}^{N}$ as $\phi_n = l_{2n-1}$; the odd-order Legendre polynomials. It is well known that the set $\{l_{2n-1}\}_{n=1}^{\infty}$ constitutes an orthonormal basis for the subspace of $L^2[-1, 1]$ consisting of functions with odd symmetry.

Fig. 1 shows the energy concentration ξ achieved by the optimal pulse (*legendre*) with $\eta = 2$ and $\eta = 3$. For small values of α, the *legendre* pulse with $\eta = 2$ is more concentrated than the *btrc* pulse by about 0.5%. The *btrc* pulse also shows a decrease in ξ as α exceeds 0.6. A comparison of the *legendre* pulse ($\eta = 3$) with the *rc* pulse reveals that the former is 1% -1.5% more concentrated than the latter when $\alpha \leq 0.5$.

We now consider the robustness of the pulses to carrier frequency offset at the receiver. We adopt the signal-to-noise and interference ratio (SINR) as a figure of merit. The mean received SINR for the system is computed by averaging (4) over all sub-carriers. For numerical simulations, K is set to 64, and we assume that $\zeta_{\delta f}$ is the uniform distribution with $0 \leq \Delta f \leq \frac{1}{2T}$. The additive Gaussian noise is such that $E_s/N_0 = 15$ dB. The mean received SINR for pulses with $\eta = 2$ is presented in Fig. 2. For a small value of the roll-off factor $\alpha = 0.2$, the *legendre* pulse achieves a gain of about 1 dB SINR. For longer pulse lengths, $\alpha = 0.8$ for instance, the margin of the SINR gain has fallen to about 0.5 dB.

The power spectrum of the OFDM signal with various shaping pulses is shown in Fig. 3. In this figure, $K = 1536$ which corresponds to TM-I mode of the DAB standard. The standard requires at least 71 dB attenuation at the sub-carrier frequency $fT = 970$. While the commonly used rectangular pulse does not meet this requirement, the *legendre* and *rc* pulses exceed the requirement with a considerable margin. The *btrc* pulse also meets the requirement, but with a much smaller margin.

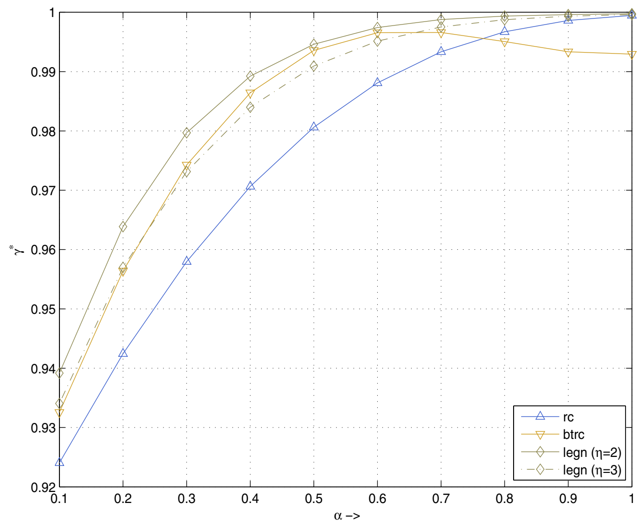

Fig. 1. Main lobe energy ratio

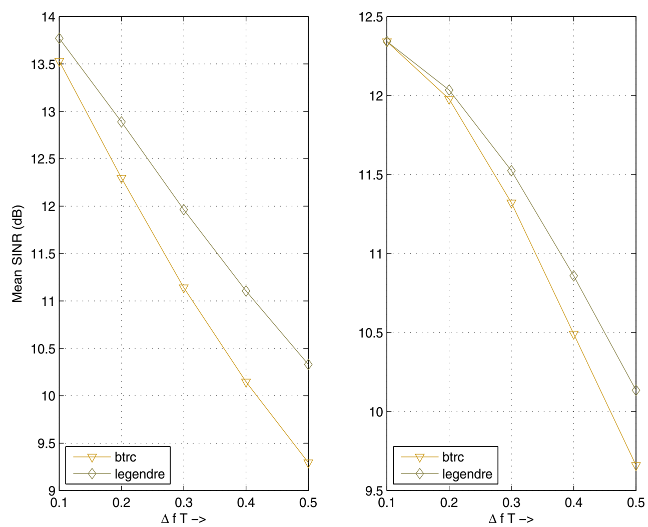

Fig. 2. Mean received SINR ($\eta = 2$, $\alpha = 0.2$, and $\alpha = 0.8$)

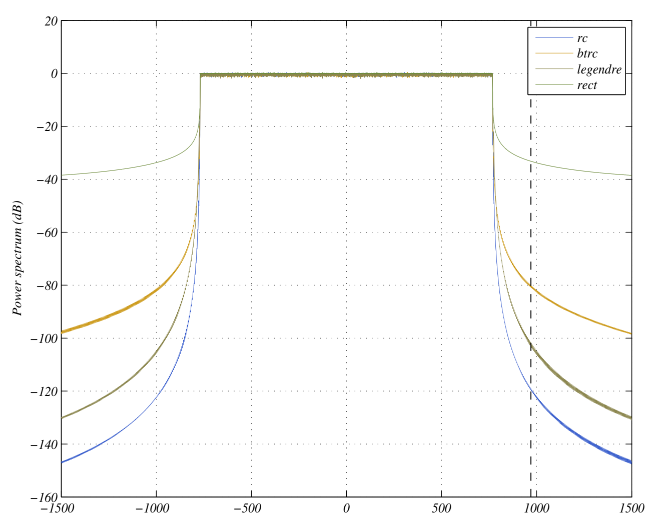

Fig. 3. Power spectra ($\alpha = 0.2$). The vertical line marks the 970th carrier.

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